Lesson 11 B:
The Petrov classification and the Weyl tensor

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April 14, 2025

As we have pointed out one of unsolved problems of General Relativity (and one that might be impossible to ever solve) is that one of detecting invariant differences between metrics. Two apparently different metrics could be representing the same manifold. It is highly non trivial to prove that a transformation of coordinates from one into the other exists or not. There are several criteria which seek to classify at least type of solutions in an invariant way. One of them is the Petrov classification.

Inspired by the fact that using the electromagnetic field strength tensor (see Lesson 5) we can define the electric and magnetic part of it:

$$E^i = F^{0i} \tag{1}$$

and

$$B^i = \frac{1}{2}\epsilon^{ijk}F_{jk} \tag{2}$$

where i, j, k run between 1 and 3.

We can now using the Weyl tensor define the following two new tensors:

$$E_{\alpha\gamma} \stackrel{\text{def}}{=} C_{\alpha\beta\gamma\delta}u^\beta u^\delta \tag{3}$$

$$H_{\alpha\gamma} \stackrel{\text{def}}{=} \frac{1}{2}\sqrt{-g}\epsilon_{\alpha\beta\mu\nu}C^{\mu\nu}{}_{\gamma\delta}u^\beta u^\delta \tag{4}$$

where $\vec{u}$ is a timelike vector and g is the determinant of the metric. These tensors just in a mathematical analogy with the electromagnetic case, are called the electric and magnetic part of the Weyl tensor. These tensors represent the Weyl tensor uniquely. Notice that the Weyl tensor as the electromagnetic tensor is a zero trace tensor. These new tensors are symmetric (although is not immediately clear from the definition), and they contain all relevant information about the Weyl tensor. We can define a new, complex, tensor:

$$Q_{\alpha\gamma} \stackrel{\text{def}}{=} E_{\alpha\gamma} + iH_{\alpha\gamma} = Q_{\gamma\alpha} \tag{5}$$

It can be easily shown that it is a tensor in 3-dimensional space.

Now this tensor can be classified according to its algebraic structure, i.e. looking at the eigenvectors and eigenvalues of it. Since its trace is zero the sum of all its eigenvalues must be zero. The classification is performed looking at the degree of the polynomial equation (i.e. the equation defined by the determinant of the condition for finding the eigenvalues equation). Weyl tensors of different Petrov types mean that the metrics are different. But two different metrics (two different space-times) can have the same Petrov type. So different Petrov types is a sufficient condition for two metric representations to be different space times but it is not a necessary condition.

1. Type I : four simple principal null directions,

2. Type II : one double and two simple principal null directions,

3. Type D : two double principal null directions,

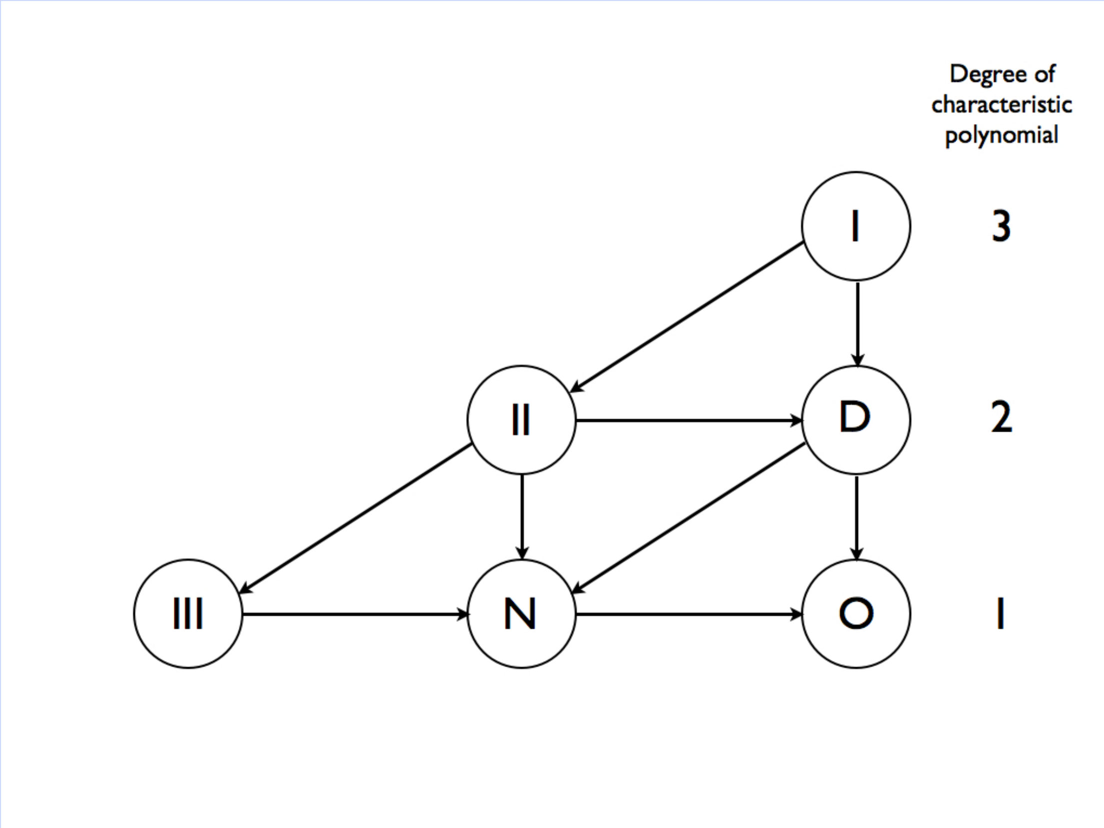
4. Type III: one triple and one simple principal null direction,

5. Type N : one quadruple principal null direction,

6. Type O : the Weyl tensor vanishes.

A type I Q tensor is called algebraically general; otherwise, it is called algebraically special . Different events in a given spacetime can have different Petrov types. The possible transitions between

Petrov types are shown in the figure, which can also be interpreted as stating that some of the Petrov types are ”more special” than others. For example, type I, the most general type, can degenerate to types II or D, while type II can degenerate to types III, N, or D.



A diagram for the Petrov classification

Physical Interpretation

1. Type D regions are associated with the gravitational fields of isolated massive objects, such as stars. The two double (repeated eigenvalues) null vectors define ”radially” ingoing and outgoing null congruences near the object which is the source of the field. Essentially these are solutions like the Kerr-Newman black-holes and represent the objects described by the No-Hair theorem. If the object is rotating about some

axis, in addition to the tidal effects, there will be various gravitomagnetic (not to be confused with what above was explained as the magnetic and electric part of the Weyl tensor effects: this is a weak-field associated with motion of the sources analogous to electric currents generated by the motion of magnetic sources in electromagnetism), such as spin-spin forces on gyroscopes carried by an observer.

2. Type III regions are associated with a kind of longitudinal gravitational radiation. In such regions, the tidal forces have a shearing effect.

This possibility is often neglected, in part because the gravitational radiation which arises in weak-field theory is type N, and in part because type III radiation decays like $O(1/r^2)$, which is faster than type N radiation.

3. Type N regions are associated with transverse gravitational radiation, which is the type we are trying to detect with LIGO. The quadruple principal null direction corresponds to the wave vector describing the direction of propagation of this radiation. It typically decays like $O(1/r)$, so the long-range radiation field is type N.

4. Type II regions combine the effects noted above for types D, III, and N, in a nonlinear way.

5. Type O regions, or conformally flat regions, are associated with places where the Weyl tensor vanishes identically.

Examples

In some well studied solutions, the Weyl tensor has the same Petrov type at each event:

1. The Kerr vacuum is everywhere type D,
2. some Robinson/Trautman vacuums are everywhere type III,
3. the pp-wave (plane-fronted waves with parallel propagation) spacetimes are everywhere type N. pp-wave solutions are of the form $ds^2 = F(u,x,y)du^2 + 2dudv + dx^2 + dy^2$ where u, v are so-called null coordinates, i.e. $u = t + z$ and $v = t - z$ and F is a smooth function.

4. the Friedman-Robertson Walker cosmological models are everywhere type O.

5. And any spherically symmetric spacetime must be of type D (or O). All algebraically special spacetimes having various types of stress-energy tensor are known, for example, all the type D vacuum solutions.

6. Some classes of solutions can be invariantly characterized using algebraic symmetries of the

Weyl tensor: for example, the class of non-conformally flat null electrovacuum or null dust solutions admitting an expanding but nontwisting null congruence is precisely the class of Robinson/Trautmann spacetimes. These are usually type II, but include type III and type N examples.

References

1. J. Plebanski & A. Krasinski, An Introduction to

General Relativity and Cosmology, Cambridge University Press (CUP), 2006.

2. H. Stephani, General Relativity, CUP, 2nd edition 1990.

3. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, Exact Solutions of Einstein's field equations, 2nd. ed, CUP 2003.

4. R. Penrose & W. Rindler, Spinors and Space-Time, CUP, 1984.