# Lesson 9
# Spherical Symmetry and Relativistic Stars

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# Spherically symmetric 4-D Riemann space

We call a Riemann manifold spherically symmetric when its isometry group is the group of rotations around a point, i.e. $O(3)$. This means that the metric should obey the Killing equations for each of its generators. The orbits of $O(3)$ are 2-dimensional spheres, and each one of them can be embedded in 3-D Euclidean space. Its equation:

$$x^2 + y^2 + z^2 = R^2 \tag{1}$$

where R is the radius of the sphere. A rotation is then described by:

$$x'^i = x^i \cos\alpha + x^j \sin\alpha, \quad x'^j = x^i \cos\alpha + x^j \sin\alpha,$$
$$x'^k = x^k \qquad \textit{for} \quad i \neq j \neq k. \tag{2}$$

The angle $\alpha$ is the group parameter. The Killing vector is:

$$\underset{[i,j]}{k^\mu} = x^j \delta^\mu{}_i - x^i \delta^\mu{}_j. \tag{3}$$

It's better to write it this way:

$$\underset{(i)}{J} \overset{def}{=} \underset{(i)}{k}^{\mu} \frac{\partial}{\partial x^{\mu}}. \tag{4}$$

And our basis will be:

$$\begin{aligned} \underset{[xy]}{J} &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \\ \underset{[yz]}{J} &= y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \\ \underset{[xz]}{J} &= x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} \end{aligned} \tag{5}$$

Let's use now spherical coordinates:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta. \tag{6}$$

Then,

$$\underset{[xy]}{J} = \frac{\partial}{\partial\phi}$$
$$\underset{[yz]}{J} = \sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}$$
$$\underset{[xz]}{J} = \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi} \tag{7}$$

We can use then $\theta$ and $\phi$ with $r$ and $t$ for the entire manifold. So we need to solve the Killing equations (see eq (86) and definition of Killing vectors in the slides from Lesson 8). This means writing eq 86 from Lesson 8 for $g_{\alpha\beta}(t, r, \theta, \phi)$, where $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$. The Killing vectors are then:

$$\underset{(1)}{k^\alpha} = \delta^\alpha{}_3$$
$$\underset{(2)}{k^\alpha} = \sin\phi\delta^\alpha{}_2 + \cos\phi\cot\theta\delta^\alpha{}_3$$
$$\underset{(3)}{k^\alpha} = \cos\phi\delta^\alpha{}_2 - \sin\phi\cot\theta\delta^\alpha{}_3 \tag{8}$$

The Killing equation for the first vector $\underset{(1)}{k}^{\alpha}$ reduces to $g_{\alpha\beta,3} = 0$.

So the metric is independent of $\phi$. For the others the Killing equations reduce to:

$$\sin\phi \frac{\partial}{\partial\theta} g_{\alpha\beta} + (\sin\phi)_{,\alpha} g_{2\beta} + (\sin\phi)_{,\beta} g_{\alpha 2} + (\cos\phi \cot\theta)_{,\alpha} g_{3\beta} + (\cos\phi \cot\theta)_{,\beta} g_{\alpha 3} = 0 \quad (9)$$

$$\cos\phi \frac{\partial}{\partial\theta} g_{\alpha\beta} + (\cos\phi)_{,\alpha} g_{2\beta} + (\cos\phi)_{,\beta} g_{\alpha 2} - (\sin\phi \cot\theta)_{,\alpha} g_{3\beta} + (\sin\phi \cot\theta)_{,\beta} g_{\alpha 3} = 0 \quad (10)$$

Now we multiply (9) by $\cos\phi$ and (10) by $\sin\phi$ and subtract one from the other. The result

$$\cos\phi(\sin\phi)_{,\alpha} g_{2\beta} + \cos\phi(\sin\phi)_{,\beta} g_{\alpha 2} + \cos\phi(\cos\phi\cot\theta)_{,\alpha} g_{3\beta} + \cos\phi(\cos\phi\cot\theta)_{,\beta} g_{\alpha 3} - \sin\phi(\cos\phi)_{,\alpha} g_{2\beta} + \sin\phi(\cos\phi)_{,\beta} g_{\alpha 2} - \sin\phi(\sin\phi\cot\theta)_{,\alpha} g_{3\beta} + \sin\phi(\sin\phi\cot\theta)_{,\beta} g_{\alpha 3} = 0 \quad (11)$$

which can be further simplified by using the identities:

$$\sin\phi(\sin\phi)_{,\alpha} + \cos\phi(\cos\phi)_{,\alpha} = 0$$
$$\cos\phi(\sin\phi)_{,\alpha} + \sin\phi(\cos\phi)_{,\alpha} \equiv \phi_{,\alpha} \quad (12)$$

Then we obtain:

$$\phi_{,\alpha} g_{2\beta} + \phi_{,\beta} g_{\alpha 2} + (\cot\theta)_{,\alpha} g_{3\beta} + (\cot\theta)_{,\beta} g_{\alpha 3} = 0 \quad (13)$$

And now we multiply (9) by $\sin\phi$ and (10) by $\cos\phi$ (reverse the order in which we did it before), and add the results instead of subtracting. Using (12) again we get:

$$\frac{\partial}{\partial\theta} g_{\alpha\beta} - \phi_{,\alpha} \cot\theta g_{3\beta} - \phi_{,\beta} \cot\theta g_{\alpha 3} = 0 \quad (14)$$

(12) is an algebraic equation, we can see that:

1. For $2 \neq \alpha \neq 3, 2 \neq \beta \neq 3$ the equation is fulfilled identically.
2. For $2 \neq \alpha \neq 3, \beta = 2$:
$$g_{\alpha 3} = 0, \qquad \alpha = 0, 1. \tag{15}$$
3. For $2 \neq \alpha \neq 3, \beta = 3$:
$$g_{\alpha 2} = 0, \qquad \alpha = 0, 1. \tag{16}$$
4. For $\alpha = 2, \beta = 2$:
$$g_{23} + g_{32} = 2g_{23} = 0 \tag{17}$$
5. For $\alpha = 2, \beta = 3$:
$$g_{33} = g_{22} \sin^2 \phi \tag{18}$$
6. For $\alpha = \beta = 3$, we get again (17).

Looking now at the differential equation (13) we get:

1. For $2 \neq \alpha \neq 3, 2 \neq \beta \neq 3$
$$\frac{\partial}{\partial \theta} g_{\alpha\beta} = 0 \qquad \alpha, \beta = 0, 1. \tag{19}$$
2. For $2 \neq \alpha \neq 3, \beta = 2$, and identity because of (15)

And...alas!

$$ds^2 = g_{00}dt^2 + 2g_{0r}drdt + g_{rr}dr^2 + F(t,r)(d\theta^2 + \sin^2\theta d\phi^2).$$

This is the more general spherically symmetric metric. Notice that it can depend of a "time-like" coordinate and a "radial" one. But we have not discussed much about coordinate restrictions given by the Killing equations. This is a non-trivial issue that I will not discuss here. But the interested student can take a look at the excellent book: An Introduction to General Relativity and Cosmology by J. Plebanski and A. Krasinski, published by Cambridge in 2006. The study of the Killing equations I presented follows to a great extent the treatment followed in this book .

Static space-times will be those in which we have:

1. $\frac{\partial}{\partial t}$ is a Killing vector, i.e. the metric components do not depend on t, and additionally,
2. the metric is invariant under time reversal, i.e. changing $t$ for $-t$.

We can take:

$$g_{00} = -e^{\nu}, \qquad g_{rr} = e^{\lambda}, \qquad F(t,r) = r^2. \tag{20}$$

And the metric becomes:

$$ds^2 = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 d\Omega^2. \tag{21}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{22}$$

Now with this

$$g^{\alpha\beta} = \begin{pmatrix} -e^{-\nu} & 0 & 0 & 0 \\ 0 & e^{-\lambda} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix} \tag{23}$$

We can now use dot for $t$ derivatives and prime for $r$ derivatives. The non vanishing components of the Einstein tensor are:

$$G_0{}^0 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{24}$$

$$G_0{}^1 = -e^{-\lambda} r^{-1}\dot{\lambda} = -e^{\lambda-\nu} G_1{}^0, \tag{25}$$

$$G_1{}^1 = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{26}$$

$$G_2{}^2 = G_3{}^3 = \frac{1}{2}e^{-\lambda}\left(\frac{\nu'\lambda'}{2} + \frac{\lambda'}{r} - \frac{\nu'}{r} - \frac{\nu'^2}{2} - \nu''\right)$$
$$+ \frac{1}{2}e^{-\nu}\left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2}\right). \tag{27}$$

The Bianchi identities reveal that (26) vanishes automatically if (23), (24) and (25) all vanish. We have then 3 equations to solve:

$$e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}=0, \tag{28}$$

$$\dot{\lambda}=0 \tag{29}$$

$$e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}=0, \tag{30}$$

Adding (27) and (29), we get

$$\lambda'+\nu'=0 \tag{31}$$

The integration gives

$$\lambda+\nu=h(t), \tag{32}$$

where h is an arbitrary function of integration.

We see that using (28) $\lambda$ is only a function of $r$ and so it is (27)

$$e^{-\lambda} - re^{-\lambda}\lambda' = 1, \tag{33}$$

which is the same as:

$$(re^{-\lambda})' = 1, \tag{34}$$

So integrating we get:

$$re^{-\lambda} = r + constant \tag{35}$$

For reasons we will make clear later we choose it to be $-2m$, obtaining:

$$e^{\lambda} = \frac{1}{1 - \frac{2m}{r}}. \tag{36}$$

To this stage the metric has been reduced to:

$$g_{\alpha\beta} = diag[-e^{h(t)}(1 - 2m/r), \\ (1 - 2m/r)^{-1}, r^2, r^2 \sin^2\theta], \tag{37}$$

To eliminate h(t) we transform to a new time $t'$

$$t' = \int_c^t e^{\frac{1}{2}h(u)} du \tag{38}$$

where c is arbitrary and ct. This will get us:

$$ds^2 = -(1 - 2m/r)dt^2 + (1 - 2m/r)^{-1}dr^2 \\ + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{39}$$

This is the famous Schwarzschild line element.

# The Schwarzschild radius

It was already realized in the 18th century that there was a particular relationship between the mass and the radius of a star for which the escape velocity would be larger than the speed of light. Using Newtonian physics we can calculate the escape velocity of an object as the speed needed to escape the Earth's gravitational attraction.

$$\frac{1}{2}mv^2 - G\frac{Mm}{r} = 0 \tag{40}$$

from this

$$v = \sqrt{2G\frac{M}{r}} \tag{41}$$

For Earth the escape velocity is $11.2km/s$.

If we look at the escape speed for a neutron star: Using $G = 6.7 \times 10^{-11} m^3/Kgs^2$ and $M = 1.4 M_\odot \approx 2.8 \times 10^{30} Kg$ and assuming a radius $r = 10Km$ we get a escape speed slightly lower than $c/2$: half the speed of light! We can see that for a radius:

$$r = \frac{2GM}{c^2} \tag{42}$$

the escape velocity is equal to the speed of light. This radius is called the Scharwzschild radius. For a star like the Sun this radius is: $r = 3 \times 10^3 m$ while its radius is $r_\odot = 7 \times 10^8 m$. When the object is a totally collapsed object the topologically spherical surface defined by the Schwarschild radius is called an event horizon.

# The precession of the perihelion

In Newtonian gravitation we have:

$$\vec{\mathsf{F}} = -m\frac{\mu}{r^2}\hat{\mathsf{r}}, \tag{43}$$

With this we get that the angular momentum is conserved, so the problem takes place in a plane and using planar polar coordinates $(R, \phi)$, the equation of motion becomes:

$$(\ddot{R} - R\dot{\phi}^2)\hat{\mathsf{R}} + \frac{1}{R}\frac{d}{dt}(R^2\dot{\phi})\hat{\phi} = -\frac{\mu}{R^2}\hat{\mathsf{R}}. \tag{44}$$

Taking the scalar product with $\hat{\mathsf{R}}$ gives:

$$\ddot{R} - R\dot{\phi}^2 = -\mu/R^2, \tag{45}$$

If we take the scalar product with $\hat{\phi}$ we see that:

$$\frac{d}{dt}(R^2\dot{\phi}) = 0 \tag{46}$$

And we see that the angular momentum is conserved:

$$R^2\dot{\phi} = h \tag{47}$$

To obtain the equation for the orbit we use $u = R^{-1}$ and we get:

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{h^2} \tag{48}$$

where $h = R^2\dot{\phi}$ is the constant angular momentum. This is called Binet's equation and is introduced in the Classical Mechanics courses. The solution is:

$$u = \frac{u}{h^2} + C\cos(\phi - \phi_0), \tag{49}$$

Where $C$ and $\phi_0$ are constants.

Going back to $R$ we get:

$$l/R = 1 + e\cos(\phi - \phi_0), \quad (50)$$

where $l = h^2/\mu$ and $e = Ch^2/\mu$. This is the polar equation of a conic curve in which $l$ is semi-latus rectum, $e$ the eccentricity, and $\phi$ the orientation respect to the $x$ axis. If $0 < e < 1$ the curve is an ellipse and the point of nearest approach to the origin is called **perihelion**. In the case of the two-body problem using the reduced mass we still get the same result even if the two bodies are comparable in size.

# The variational method for geodesics

I will present a treatment for the calculation of geodesics that could be helpful in some instances, but also sheds light on the geometrical meaning of geodesics. We define first a Lagrangian functional:

$$L = L(x^a, \dot{x}^a, u), \tag{51}$$

where $u$ is a parameter along the curve, and the dot signifies derivation respect to u. We will define this functional as:

$$L = [g_{ab}(x)\dot{x}^a\dot{x}^b]^{\frac{1}{2}}. \tag{52}$$

We the define the action:

$$\int_{P_1}^{P_2} L du = \int_{P_1}^{P_2} ds = s, \tag{53}$$

where $s$ is the interval between two arbitrary point $P_1$ and $P_2$ on a curve connecting them.

The geodesics is defined as the curve joining these two points whose interval is an extremum. So we need to solve for $\delta s = 0$
The solution is given by Euler-Lagrange equations:

$$\frac{\partial L}{\partial x^a} - \frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = 0. \tag{54}$$

To avoid square roots:

$$2L\left[\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a}\right] = 0. \tag{55}$$

which can be written:

$$\frac{d}{du}\left(\frac{\partial L^2}{\partial \dot{x}^a}\right) - \frac{\partial L^2}{\partial x^a} = 2\frac{\partial L}{\partial \dot{x}^a}\frac{dL}{du}. \tag{56}$$

Using (51) we get:

$$\begin{aligned}\frac{d}{du}\left(\frac{\partial L^2}{\partial \dot{x}^a}\right)-\frac{\partial L^2}{\partial x^a} &= \frac{d}{du}\left[\frac{\partial}{\partial \dot{x}^a}(g_{bc}\dot{x}^b\dot{x}^c)\right]-\frac{\partial}{\partial x^a}(g_{bc}\dot{x}^b\dot{x}^c)\\ &= \frac{d}{du}(2g_{ab}\dot{x}^b)-\frac{\partial g_{bc}}{\partial x^a}\dot{x}^b\dot{x}^c\\ &= 2g_{ab}\ddot{x}^b+2\frac{\partial g_{ab}}{\partial x^c}\dot{x}^b\dot{x}^c-\frac{\partial g_{bc}}{\partial x^a}\dot{x}^b\dot{x}^c\\ &= 2g_{ab}\ddot{x}^b+2\dot{x}^b\dot{x}^c\Gamma_{bca} \qquad (57)\end{aligned}$$

and the right hand side of (56) yields:

$$\begin{aligned}2\frac{\partial L}{\partial \dot{x}^a}\frac{dL}{du} &= 2\frac{\partial}{\partial \dot{x}^a}(g_{bc}\dot{x}^b\dot{x}^c)^{1/2}\frac{d}{du}\left(\frac{ds}{du}\right)\\ &= 2(g_{bc}\dot{x}^b\dot{x}^c)^{-1/2}g_{ad}\dot{x}^d\frac{d^2s}{du^2}\\ &= 2\left(\frac{d^2s}{du^2}\Big/\frac{ds}{du}\right)g_{ab}\dot{x}^b. \qquad (58)\end{aligned}$$

Equating (56) and (57) and choosing the parameter $u = s$ (57) is zero and we get:

$$\ddot{x}^b + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0 \tag{59}$$

which we knew from before, but now we know that we can define the quantity:

$$2K \equiv g_{ab}(x)\dot{x}^a\dot{x}^b = \alpha, \tag{60}$$

such that:

$$\frac{\partial K}{\partial x^a} - \frac{d}{du}\left(\frac{\partial K}{\partial \dot{x}^a}\right) = 0 \tag{61}$$

and we require $2K$:

$$2K = \alpha = \begin{cases} 0, \\ +1, \\ -1, \end{cases} \tag{62}$$

What we will do now is look at the geodesics of the Schwarzschild metric. And we will use from the previous paragraph:

$$2K = -(1 - 2m/r)\dot{t}^2 + (1 - 2m/r)^{-1}\dot{r}^2 \\ + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2 = -1 \tag{63}$$

Deriving the Euler-Lagrange equations from (59), the three simplest ones correspond to $a = 0, 2, 3$ in (60):

$$\frac{d}{d\tau}[(1 - 2m/r)\dot{t}] = 0, \tag{64}$$

$$\frac{d}{d\tau}(r^2\dot{\theta}) - r^2 \sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{65}$$

$$\frac{d}{d\tau}(r^2 \sin^2\theta\dot{\phi}) = 0. \tag{66}$$

we need four equations to find the four unknown,
$t = t(\tau), \quad r = r(\tau), \quad \theta = \theta(\tau), \quad \phi = \phi(\tau)$, and (63)-(65) plus (62) provide them. First we can see by picking $\theta = \frac{1}{2}\pi$ that $\dot{\theta} = 0$ and higher derivatives as well. This means that motion occurs in a plane like with Newtonian gravity. So we get:

$$r^2\dot{\phi} = h \tag{67}$$

where $h$ is a constant (remember (47)). The same manner (63) yields:

$$(1 - 2m/r)\dot{t} = k, \tag{68}$$

where $k$ is a constant, and substituting in (62)

$$k^2(1 - 2m/r)^{-1} - (1 - 2m/r)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 1 \tag{69}$$

We do the same we did with the classical theory and make $u = 1/r$, which yields:

$$\frac{dr}{d\phi} = \frac{dr}{dt}\frac{1}{\dot{\phi}} = -\frac{1}{u^2}\frac{du}{d\phi},$$
$$\dot{r} = -\dot{\phi}r^2\frac{du}{d\phi} = -h\frac{du}{d\phi} \tag{70}$$

Then using (66) we get from (69),

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{k^2 - 1}{h^2} + \frac{2m}{h^2}u + 2mu^3. \tag{71}$$

This is a first order non-linear differential equation. It can be integrated using elliptical functions.

But we can differentiate respect to $\phi$ once more and get:

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2. \tag{72}$$

We can compare it with the Binet equation from Classical mechanics:

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{h^2} \tag{73}$$

and see that in the relativistic version we have an extra term $3m/r^2$. The ratio of the two terms is:

$$\frac{3mu^2}{m/h^2} = \frac{3h^2}{r^2} \tag{74}$$

And this quantity for Mercury is $\approx 10^{-7}$ (and quite much smaller for all the other planets!).

We will solve equation (72) by a perturbation scheme. We define a parameter ($c = 1$):

$$\epsilon = 3m^2/h^2, \tag{75}$$

Now we get ($' = d/d\phi$):

$$u'' + u = \frac{m}{h^2} + \epsilon \left( \frac{h^2 u^2}{m} \right). \tag{76}$$

And assume the solution is

$$u = u_0 + \epsilon u_1 + O(\epsilon^2). \tag{77}$$

We substitute and find:

$$u_0'' + u_0 - \frac{m}{h^2} + \epsilon \left( u_1'' + u_1 - \frac{h^2 u_0^2}{m} \right) + O(\epsilon^2) = 0 \tag{78}$$

We make each power of $\epsilon$ equal to zero. To zero order we recover the classical solution:

$$u_0 = \frac{m}{h^2}(1 + e\cos\phi), \tag{79}$$

where we took $\phi_0 = 0$. The first order:

$$\begin{aligned} u_1'' + u_1 &= \frac{m}{h^2}(1 + e\cos\phi)^2 \\ &= \frac{m}{h^2}(1 + 2e\cos\phi + e^2\cos^2\phi) \\ &= \frac{m}{h^2}(1 + \frac{1}{2}e^2) + \frac{2me}{h^2}\cos\phi + \frac{me^2}{2h^2}\cos 2\phi \end{aligned} \tag{80}$$

where we used that $2\cos^2\phi = 1 + \cos 2\phi$, And we try:

$$u_1 = A + B\phi\sin\phi + C\cos 2\phi, \tag{81}$$

Then we get:

$$A = \frac{m}{h^2}(1 + \frac{1}{2}e^2), \quad B = \frac{me}{h^2}, \quad C = -\frac{me^2}{6h^2}, \tag{82}$$

And the general solution becomes:

$$u \simeq u_0 + \epsilon \frac{m}{h^2}[1 + e\phi \sin\phi + e^2(\frac{1}{2} - \frac{1}{6}\cos 2\phi)]. \tag{83}$$

The most important term in the correction is $e\phi \sin\phi$ because it grows which each orbit. So we keep:

$$u \simeq \frac{m}{h^2}[1 + e\cos\phi + \epsilon e\phi \sin\phi], \tag{84}$$

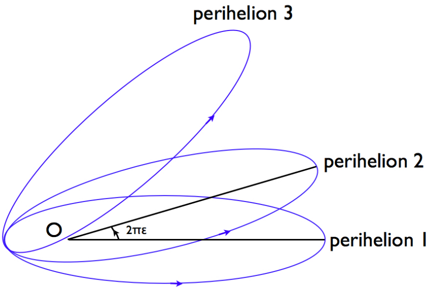
or neglecting terms of order $\epsilon^2$ can be put:

$$u \simeq \frac{m}{h^2}\{[1 + e\cos[\phi(1-\epsilon)]\}, \tag{85}$$

So the orbit is now approximately an ellipse but the period is not $2\pi$, it is:

$$\frac{2\pi}{1-\epsilon} \simeq 2\pi(1+\epsilon). \tag{86}$$

The planet travels in an ellipse but with its axis rotating, moving on by an amount $2\pi\epsilon$ between the two points of closest approach.



In non-relativistic units it becomes:

$$2\pi\epsilon \simeq \frac{24\pi^3 a^2}{c^2 T^2(1-e^2)}, \tag{87}$$

where $a$ is the semi-major axis of the ellipse and $T$ is the period of the orbit. Although the precession is not necessarily a relativistic phenomenon -perturbation from the other planets has these effects as well- only general relativity explains the one in Mercury.
In what follows there is a table with the predicted GR precession for other bodies.

One interesting case is 1566 Icarus. 1556 Icarus is an Apollo asteroid (a sub-class of near-Earth asteroid) whose unusual characteristic is that at perihelion it is closer to the Sun than Mercury (it is said to be a Mercury-crosser asteroid) . At perihelion is at 0.187 AU from the Sun while aphelion is about 2 AU. Its precession was measured in 1971.

Theoretical and Observational values of residual precession

| **Planet** | **GR prediction** | **Observed** |
|---|---|---|
| Mercury | 43.0 | $43.1 \pm 0.5$ |
| Venus | 8.6 | $8.4 \pm 4.8$ |
| Earth | 3.8 | $5.0 \pm 1.2$ |
| Icarus | 10.3 | $9.8 \pm 0.8$ |

I will not treat the bending of light, the other major specific prediction of GR which can be calculated using the Schwarschild metric.

Static stars mean the fluid has no motion. So the energy -momentum tensor is:

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu}, \tag{88}$$

The only non-zero component of $\vec{U}$ is $U^0$ which because of the normalization

$$\vec{U} \cdot \vec{U} = -1 \tag{89}$$

implies:

$$U^0 = e^{-\nu/2}, \qquad U_0 = -e^{\nu/2} \tag{90}$$

where I have used the metric (20).

$T$ then will have components:

$$
\begin{aligned}
T_{00} &= \rho e^{\nu}, \\
T_{rr} &= p e^{\lambda}, \\
T_{\theta\theta} &= r^2 p, \\
T_{\phi\phi} &= \sin^2\theta \, T_{\theta\theta},
\end{aligned}
\tag{91}
$$

All others vanish. We expect that the equation of state due to thermodynamic equilibrium (static solution) there will be a solution:

$$
p = p(\rho, S) \tag{92}
$$

Furthermore we can assume that $p$ only depends on $\rho$.

The conservation laws $T^{\alpha\beta}{}_{;\beta} = 0$ imply just one equation, due to symmetry:

$$(\rho + p)\frac{d\nu}{dr} = -\frac{dp}{dr}. \quad (93)$$

This equation tells what should be the pressure gradient to sustain the star in equilibrium.
Now looking at Einstein equations we will replace $\lambda(r)$ with a function $m(r)$

$$m(r) = \frac{r}{2}(1 - e^{-\lambda}), \quad (94)$$

So:

$$g_{rr} = e^{\lambda} = \frac{1}{1 - \frac{2m(r)}{r}} \quad (95)$$

The (0, 0) component of Einstein's eqs. gives:

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho. \quad (96)$$

We will need to explore the meaning of this $m(r)$ carefully because energy is not localizable. From $G_{rr} = 8\pi T_{rr}$ we get,

$$-\frac{1}{r^2}e^{\lambda}(1 - e^{-\lambda}) + \frac{2}{r}\frac{d\nu}{dr} = 8\pi p e^{\lambda} \tag{97}$$

and

$$\frac{d\nu}{dr} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]} \tag{98}$$

We can see that if we have an equation of state $p = p(\rho)$ then this equation with (92), (95) and (97) are four equations for the four unknowns $\nu, m, \rho, p$. This fully determines the problem.

# The exterior geometry

In the region outside the star we have $\rho = p = 0$, and we get,

$$\frac{dm}{dr} = 0, \tag{99}$$

$$\frac{d\nu}{dr} = \frac{m}{r(r-2m)}. \tag{100}$$

which have solutions, if we require that $\nu \to 0$ as $r \to \infty$:

$$m(r) = M = const., \tag{101}$$

$$e^{\nu} = 1 - \frac{2M}{r}. \tag{102}$$

which gives our old friend,the Schwarzschild metric:

$$ds^2 = -(1-2m/r)dt^2 + (1-2m/r)^{-1}dr^2 + r^2 d\Omega^2, \tag{103}$$

which for large $r$ becomes:

$$ds^2 = -(1-2m/r)dt^2 + (1+2m/r)dr^2 + r^2 d\Omega^2, \tag{104}$$

# Birkhoff's theorem

*(Without proof)*

The Schwarzschild metric is the only spherically symmetric, asymptotically flat solution to Einstein's vacuum field equations.

**The interior structure of a star**

Dividing equation (92) by $\rho + p$ and eliminating $\nu$ from (97), we obtain the **Oppenheimer-Volkov** equation,

$$\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)}, \qquad (105)$$

Combine with equation (95) and an equation of state this gives three equations for the three unknowns $m, \rho, p$.

Notice that we can get $\nu$ once we solve this three equations, just from (93). In the integration process we will need to check carefully matching conditions at the surface and reasonable initial conditions. I f we look at the Newtonian limit of the fluid equations we consider $p \ll \rho$ which means $4\pi r^3 p \ll m$ Because we expect the metric to be nearly flat we also require $m \ll r$. So equation (105) becomes:

$$\frac{dp}{dr} = -\frac{\rho m}{r^2} \tag{106}$$

This is the same equation for hydrostatic equilibrium in Newtonian stars.

We can assume $\rho = const$ Note that speed of sound $(dp/d\rho)^{1/2}$ is infinite! We can integrate (96) right away,

$$m(r) = 4\pi\rho r^3/3, \qquad r \leqslant R, \tag{107}$$

where $R$ is the undetermined star radius. Continuity means:

$$m(r) = 4\pi\rho R^3/3 = M, \qquad r \geqslant R, \tag{108}$$

where $M$ is the Schwarzschild mass. Solving the O-V equations (105)

$$\frac{dp}{dr} = -\frac{4}{3}\pi r \frac{(\rho + p)(\rho + 3p)}{1 - 8\pi r^2\rho/3} \tag{109}$$

This can be integrated from an arbitrary central pressure $p_c$, to give,

$$\frac{\rho + 3p}{\rho + p} = \frac{\rho + 3p_c}{\rho + p_c}\left(1 - 2\frac{m}{r}\right)^{1/2} \tag{110}$$

From which we can see that:

$$R^2 = \frac{3}{8\pi\rho}\left[1 - \frac{(\rho + p_c)^2}{(\rho + 3p_c)^2}\right] \tag{111}$$

$$p_c = \rho[1 - (1 - 2M/R)^{1/2}]/[3(1 - 2M/R)^{1/2} - 1]. \tag{112}$$

Replacing $p_c$ in (110):

$$p_c = \rho\frac{(1 - 2Mr^2/R^3)^{1/2} - (1 - 2M/R)^{1/2}}{3(1 - 2M/R)^{1/2} - (1 - 2Mr^2/R^3)^{1/2}} \tag{113}$$

and using the boundary conditions at $r = R$

$$e^{\nu} = \frac{3}{2}(1 - 2M/R)^{1/2} - \frac{1}{2}(1 - 2Mr^2/R^3)^{1/2}, \quad r \leqslant R \tag{114}$$

$$e^{\lambda} = (1 - r^2/R^2)^{-1}, \quad r \leqslant R \tag{115}$$

Notice that eq. (112) implies $p_c \to \infty$ as $M/R \to 4/9$.
This is a very general limit for $M/R$ even for truly realistic stars.

Buchdal (1981) found a solution for the equation of state:

$$\rho = 12(p_* p)^{1/2} - 5p, \tag{116}$$

where $p_*$ is an arbitrary constant. It has two properties:

- causality: demand $(dp/d\rho)^{1/2} < 1$;
- for small $p$ it reduces to

$$\rho = 12(p_* p)^{1/2}, \tag{117}$$

where $p_*$ is an arbitrary constant, which in Newtonian theory is an $n = 1$ polytrope. In Astrophysics a polytrope is a solution of the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{118}$$

where

$$\xi = r\left(\frac{4\pi G\rho_c^2}{(n+1)P_c}\right)^{\frac{1}{2}} \tag{119}$$

and

$$\rho = \rho_c\theta^n, \tag{120}$$

where $c$ refers to values of pressure and density at the center of the sphere. The polytropic equation of state is:

$$p = K\rho^{(1+\frac{1}{n})}, \tag{121}$$

where $K$ is a constant. Causality then demands additionally:

- $p < p_*$,
- $\rho < 7p_*$

It is customary to introduce another free parameter defining a new radial coordinate $r'$:

$$u(r') := \beta \frac{\sin Ar'}{Ar'}, \qquad A^2 := \frac{288\pi p_*}{1-2\beta}, \tag{122}$$

And then,

$$r(r') = r' \frac{1-\beta+u(r')}{1-2\beta}, \tag{123}$$

You can check the complete solution in Schutz' book.
But one important thing to notice is that $\beta$ is the value of $M/R$ on the surface of the star.

This justifies backtracking to the physical relevance of the metric terms.
The metric is independent of the time, meaning that any particle following a geodesic has constant momentum $p_0$, which we can define to be $p_0 = -E$. But of course a local inertial observer at rest instantaneously at any radius $r$ of S-T measures a different energy. The four velocity has to be $U^i = dx^i/d\tau = 0$ and $\vec{U} \cdot \vec{U} = 1$ implies $U^0 = e^{-\nu}$. And the energy measured will then be:

$$E = -\vec{U} \cdot \vec{p} = e^{-\nu} E \tag{124}$$

So $E$ is the energy that a distant observer will measure if the particle is far away.

Let's look at a photon emitted at radius $r_1$ and received very far away. If its frequency in the local inertial frame is $\nu_{em}$, then the local energy will be $h\nu_{em}$ and the associate conserved quantity is: $h\nu_{em}e^{\nu(r_1)}$. Then the redshift will be:

$$z = \frac{\lambda_{rec} - \lambda_{em}}{\lambda_{em}} = \frac{\nu_{em}}{\nu_{rec}} - 1 \qquad (125)$$

will be:

$$z = e^{-\nu(r_1)} - 1 \qquad (126)$$

Then back to Buchdal's solution we see that the surface redshift of the star is:

$$z = (1 - 2\beta)^{-1/2} - 1 \qquad (127)$$

It can be shown (see Schutz) that reasonable physical demands forces

$$0 < \beta < \frac{1}{6} \tag{128}$$

This range covers physically reasonable models, from Newtonian ($\beta \approx 0$) to a very relativistic surface redshift of 0.22. It was mentioned before that there can be no uniform density stars with radii smaller than $9/4M$. To support this radii it will require pressures larger than infinite! This is *Buchdal's Theorem*. If we are at the limit of this configuration any extra pressure inwards would make it collapse. The interior solution will be vacuum and we have a black hole.