Lesson 8
Gravitational Radiation

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1 Gauge transformations

A type of coordinates that leave equation (1) with the condition $|h_{\alpha\beta}| \ll 1$ unchanged is a small change in the coordinates of the form:

$$x^{\alpha'} = x^\alpha + \xi^\alpha(x^\beta), \tag{1}$$

We assume ξ^α is small in the sense that $|\xi^\alpha{}_{,\beta}| \ll 1$

$$\Lambda^{\alpha'}{}_\beta = \frac{\partial x'}{\partial x^\beta} = \delta^\alpha{}_\beta + \xi^\alpha{}_{,\beta}, \tag{2}$$

$$\Lambda^\alpha{}_{\beta'} = \delta^\alpha{}_\beta - \xi^\alpha{}_{,\beta} + O(|\xi^\alpha{}_{,\beta}|^2). \tag{3}$$

To first order this gives:

$$g_{\alpha'\beta'} = \eta_{\alpha\beta} + h_{\alpha\beta} - \xi_{\alpha,\beta} + \xi_{\beta,\alpha}, \tag{4}$$

where:

$$\xi_\alpha = \eta_{\alpha\beta}\xi^\beta \tag{5}$$

The effect of this change of coordinates is to change $h_{\alpha\beta}$:

$$h_{\alpha\beta} \to h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}, \tag{6}$$

If $|\xi^\alpha{}_{,\beta}| \ll 1$ then the new $h_{\alpha\beta}$ is also small. A change like this is called: a **gauge transformation**. This freedom of the Einstein's theory is extremely useful and important.

The Riemann tensor:
Using (1) the calculation yields:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}) \tag{7}$$

The trick or magic: i.e. the fact that the Riemann tensor does not depend on the small quantities $\xi_{\alpha,\beta}$ is precisely the fact that they are second order, and we don't need to consider them at first order.

Weak-field Einstein equations
We will define

$$h^{\mu}_{\beta} := \eta^{\mu\alpha} h_{\alpha\beta}, \tag{8}$$

$$h^{\mu\nu} := \eta^{\nu\beta} h^{\mu}_{\beta}, \tag{9}$$

the trace:

$$h := h^{\alpha}_{\alpha} \tag{10}$$

and another tensor called the "trace reverse" of $h_{\alpha\beta}$

$$\bar{h}^{\alpha\beta} := h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} h. \tag{11}$$

The trace is:

$$\bar{h} := \bar{h}^{\alpha}_{\alpha} = h^{\alpha\beta}\eta_{\beta\alpha} - \frac{1}{2}\eta^{\alpha\beta}\eta_{\beta\alpha} h = h - \frac{4}{2}h = -h \tag{12}$$

and the inverse of (19) is the same equation:

$$h^{\alpha\beta} = \bar{h}^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}\bar{h}. \tag{13}$$

The Einstein tensor becomes:

$$G_{\alpha\beta} = -\frac{1}{2}[\bar{h}_{\alpha\beta,\mu}{}^{,\mu} + \eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{,\mu\nu} - \bar{h}_{\alpha\mu,\beta}{}^{,\mu} \tag{14}$$

$$-\bar{h}_{\beta\mu,\alpha}{}^{,\mu} + O(h^2_{\alpha\beta})]. \tag{15}$$

The Lorentz gauge
Things would be simpler if we required:

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0. \tag{16}$$

Notice, that from the definition of $\bar{h}^{\alpha_\beta}$ in eq. (19), we get:

$$h^{\alpha\beta}{}_{,\beta} - \frac{1}{2}h_{,\alpha} = 0. \tag{17}$$

We just need to choose coordinates where these equations, (24) and (25), would be satisfied.

Eq (24) is called the Lorentz gauge. Let's assume we have an $\bar{h}^{\alpha_\beta}$ for which this does not hold. We look for a new one:

$$\bar{h}^{(new)}_{\mu\nu} = \bar{h}^{(old)}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha} \tag{18}$$

The divergence of this new $\bar{h}^{(new)}_{\mu\nu}$ will be:

$$\bar{h}^{(new)\mu\nu}{}_{,\nu} = \bar{h}^{(old)\mu\nu}{}_{,\nu} - \xi^{\mu,\nu}{}_{,\nu}. \tag{19}$$

But then all we need is:

$$\Box\xi^{\mu} = \xi^{\mu,\nu}{}_{,\nu} = \bar{h}^{(old)\mu\nu}{}_{,\nu} \tag{20}$$

Notice that the $\Box$ operator above is nothing else than the standard Laplacian operator minus the second time derivative, i.e. for any function f:

$$\Box f = f^{,\mu}{}_{,\mu} = \left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) f. \tag{21}$$

The inhomogeneous form of this equation always have a solution provided the function that makes the equation inhomogeneous is "well behaved". But the solution will be defined up to any other function η which satisfies:

$$\Box\eta^{\mu} = 0 \tag{22}$$

This function will of course satisfy equation (28). This procedure defines rather than a gauge, a class of gauges. In this gauge:

$$G^{\alpha\beta} = -\frac{1}{2}\Box\bar{h}^{\alpha\beta} \tag{23}$$

And consequently the weak field Einstein equations become:

$$\Box\bar{h}^{\mu\nu} = -16\pi T^{\mu\nu} \tag{24}$$

These are also called the field equations in the "linearized" theory.

But notice that in vacuum these equations are:

$$G^{\alpha\beta} = \Box\bar{h}^{\alpha\beta} = 0 \tag{25}$$

But this means that if we assume a time dependence for the metric perturbation $\bar{h}^{\alpha\beta}$ it does satisfy the wave equation if its a solution of Einstein's equations!

We have seen that if we are far enough from the sources and assume that the field is weak, we can reduce the metric to a form like the one in equation (1) Lesson 9:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{26}$$

where $|h_{\alpha\beta}| \ll 1$ everywhere, and $\eta_{\alpha\beta}$ is the flat Minkowski metric. We also proved that by a judicious choice of gauge we can arrive at the following form of Einstein's vacuum equations:

$$G^{\alpha\beta} = -\frac{1}{2}\Box\bar{h}^{\alpha\beta} = 0 \tag{27}$$

or more explicitly:

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}^{\alpha\beta} = 0 \tag{28}$$

This is the wave equation. We will explore the implications of Einstein's equations being reduced to this form. We can assume a solution of the form:

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} \exp(ik_\alpha x^\alpha), \tag{29}$$

where $\{k_\alpha\}$ is a real one form and $\{A^{\alpha\beta}\}$ some tensor whose components -constant- could be complex. Equation (1) can be written:

$$\eta^{\mu\nu} \bar{h}^{\alpha\beta}{}_{,\mu\nu} = 0, \tag{30}$$

Using (2) we get:

$$\bar{h}^{\alpha\beta}{}_{,\mu} = ik_\mu \bar{h}^{\alpha\beta} \tag{31}$$

and then (3) :

$$\eta^{\mu\nu} \bar{h}^{\alpha\beta}{}_{,\mu\nu} = -\eta^{\mu\nu} k_\mu k_\nu \bar{h}^{\alpha\beta} = 0, \tag{32}$$

and this can only vanish if

$$\eta^{\mu\nu} k_\mu k_\nu = k^\nu k_\nu = 0, \tag{33}$$

This means k is a null vector, tangent to the world line of a photon (a Minkowkski null vector). From the definition (4) we can see that if $k_\alpha x^\alpha$ is constant then $\bar{h}^{\alpha\beta}$ is constant:

$$k_\alpha x^\alpha = k_0 t + \mathbf{k} \cdot \mathbf{x} = const, \tag{34}$$

where $\mathbf{k}$ refers to the spatial components of the coordinates. The Bianchi identities require:

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0, \tag{35}$$

We can see using equation (6) that this immediately translates in:

$$\begin{aligned} \bar{h}^{\alpha\beta}{}_{,\beta} = ik_\beta \bar{h}^{\alpha\beta} = ik_\beta A^{\alpha\beta} \exp(ik_\alpha x^\alpha) \\ = k_\beta A^{\alpha\beta} = 0 \end{aligned} \tag{36}$$

This implies that $A^{\alpha\beta}$ is orthogonal to $\vec{k}$. Usually the k^0 component of the vector is called the frequency of the wave ω, so,

$$\vec{k} \rightarrow (\omega, \mathbf{k}) \tag{37}$$

We can parametrize the curve to which $\vec{k}$ is tangent in this manner:

$$x^\mu(\lambda) = k^\mu \lambda + l^\mu, \tag{38}$$

where λ is a parameter and l^μ is a constant vector (at $\lambda = 0$ is the position of a photon traveling along this line. If we calculate

$$k_\mu x^\mu(\lambda) = k_\mu k^\mu \lambda + k_\mu l^\mu = k_\mu l^\mu = const., \tag{39}$$

This means that the gravitational wave travels, like a photon, at the speed of light, and $\vec{k}$ is the direction of travel. The fact that $\vec{k}$ is null means:

$$\omega^2 = |\mathbf{k}|^2, \tag{40}$$

which is referred at the dispersion relation for the wave. So the phase velocity:

$$\frac{\omega^2}{|\mathbf{k}|^2} = 1, \tag{41}$$

and the group velocity also:

$$\frac{\partial \omega}{\partial k} = 1, \tag{42}$$

So the solution (4) $A^{\alpha\beta} \exp(ik_\alpha x^\alpha)$ to our vacuum Einstein's equation is a wave, a plane wave. Any general solution of equation (3) will be a superposition of plane waves.

The transverse-traceless gauge

It would be good if we could characterize the nature of the plane gravitational wave further: i.e. it would be good if we could estimate what are the actual measurable physical parametere and what are the effects of those for a physical observer. Can we use our gauge freedom to find any further restriction than just (11)? Any vector solution of (3), namely a ξ_α,

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) \xi_\alpha = 0 \tag{43}$$

can we use to see if we can "refine" the physical parameters of our solution better. We can choose:

$$\xi_\alpha = B_\alpha \exp(ik_\mu x^\mu), \tag{44}$$

where B_α is a constant and k^μ is the same one as before. This would give the following change in our original solution:

$$h_{\alpha\beta}^{NEW} = h_{\alpha\beta}^{OLD} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{45}$$

which give a new

$$\bar{h}_{\alpha\beta}^{NEW} = \bar{h}_{\alpha\beta}^{OLD} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}\xi^\mu_{,\mu}, \tag{46}$$

$$\bar{h}_{\alpha\beta}^{NEW} = \bar{h}_{\alpha\beta}^{OLD} - iB_\alpha k_\beta - iB_\beta k_\alpha + i\eta_{\alpha\beta} B^\mu k_\mu. \tag{47}$$

and we can choose B_α so

$$A^\alpha_{\ \alpha} = 0 \tag{48}$$

and

$$A_{\alpha\beta} U^\beta = 0 \tag{49}$$

where $\vec{U}$ is a fixed four-velocity. Equations (11),

$$A^{\alpha\beta}k_{\beta} = 0 \tag{50}$$

(23) and (24) constitute together what is called the transverse-traceless (TT) gauge conditions. Traceless because of equation (23) and transverse because we'll see that equation (25) means that the perturbation -i.e. the wave- is transversal to the direction of propagation. In passing we should mention that all the constraints utilized make our $\bar{h}$ from equation (4) is now the same as the original h in the TT gauge.

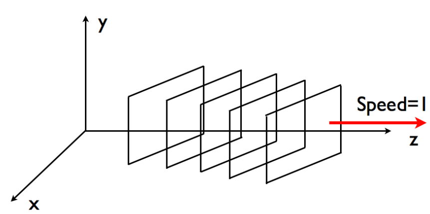
$$\bar{h}^{TT}_{\alpha\beta} = h^{TT}_{\alpha\beta} \tag{51}$$

What now remains is physically significant. We can choose $\vec{U}$ such that $U^{\beta} = \delta^{\beta}{}_{0}$ Then equation (24) means:

$$A_{\alpha\beta}U^{\beta} = A_{\alpha\beta}\delta^{\beta}{}_{0} = A_{\alpha 0} = 0 \tag{52}$$

and this for all α. In this frame we can we can orient the space coordinate axis such that the space components$\vec{k}$ is along the z direction, $\vec{k} \to (\omega, 0, 0, \omega)$ and $A_{\alpha z} = 0$ This means that $A_{\alpha\beta}$ should be of this form:

$$(A^{TT}_{\alpha\beta}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{53}$$



Plane waves traveling in the z direction

The geodesic equation

Let's look at a particle that is in the background of this plane gravitational wave. The particle has velocity U^{α} such that the geodesic equation:

$$\frac{d}{d\tau}U^{\alpha} + \Gamma^{\alpha}{}_{\mu\nu}U^{\mu}U^{\nu} = 0 \tag{54}$$

Before the wave arrives the particle is at rest, so the initial value of the acceleration is:

$$\frac{d}{d\tau}U^{\alpha} \mid_{0} = -\Gamma^{\alpha}{}_{00} = -\frac{1}{2}\eta^{\alpha\beta}(h_{\beta 0,0} + h_{0\beta,0} - h_{00,\beta}) \tag{55}$$

The particle remains forever at rest?! Yes but only in this coordinate system: the TT gauge is a system that is attached to the coordinates. If we have to particles, one of them at $x = 0$ and the other at $x = \epsilon$, both at $y = z = 0$, the proper distance:

$$\begin{aligned} \Delta l \equiv \int |ds^2|^{1/2} &= \int |g_{\alpha\beta} dx^\alpha dx^\beta|^{1/2} \\ = \int_0^\epsilon |g_{xx}|^{1/2} dx &\approx |g_{xx}(x=0)|^{1/2} \epsilon \\ &\approx \left[1 + \frac{1}{2} h_{xx}^{TT}(x=0)\right] \epsilon \end{aligned} \tag{56}$$

We can see that the proper distance changes with time. An interesting Lesson regarding the meaning of coordinates in General Relativity:

1. Coordinates have little significance per se.
2. Coordinate dependent quantities should be taken with a grain of salt.
3. An example of this is the ”position” of a particle.
4. Coordinate independent numbers are the relevant ones and are the ones that in general contain physical information.
5. Proper distance is an example.

The equation of geodesic deviation: tidal forces

Let’s look at two neighboring particles that are moving along geodesics of the gravitational wave. Equation (61) from Lesson 7:

$$\nabla_V(\nabla_V \xi^a) = R^a{}_{bcd} V^b V^c \xi^d \tag{57}$$

can be written in terms of a vector $\vec{U}$ tangent to the geodesics of the spacetime ”generated” by the passing gravitational wave.

$$\nabla_U(\nabla_U \xi^a) = R^a{}_{bcd} U^b U^c \xi^d \tag{58}$$

If we write in this in the inertial local frame (we are not using the coordinates of the TT gauge), where the connection coefficients vanish and then covariant derivatives are just regular ones:

$$\frac{d^2}{d\tau^2} \xi^a = R^a{}_{bcd} U^b U^c \xi^d \tag{59}$$

$\vec{U} = d\vec{x}/d\tau$ is the four velocity of the two particles. We need to work only up to first order in the coordinates... With this $\vec{U} \to (1, 0, 0, 0)$ and $\vec{\xi} \to (0, \epsilon, 0, 0)$ and equation (34) becomes

$$\frac{d^2}{d\tau^2} \xi^a = \frac{\partial^2}{\partial \tau^2} \xi^a = \epsilon R^a{}_{00x} = -\epsilon R^a{}_{0x0} \tag{60}$$

We can now attempt to an interpretation of equation (35). The Riemann tensor is gauge invariant, and $\vec{\xi}$, the connecting vector between the two particles, gives the proper lengths of the vector that ”measures” the distance between the two particles, and the right hand side tells exactly how these are changing. We can write the components of this equation in the TT gauge, but we know that the result is independent of the gauge:

$$
\begin{aligned}
R^x{}_{0x0} &= R_{x0x0} = \frac{1}{2}h^{TT}_{xx,00}. \\
R^y{}_{0x0} &= R_{y0x0} = \frac{1}{2}h^{TT}_{xy,00}. \\
R^y{}_{0y0} &= R_{y0y0} = \frac{1}{2}h^{TT}_{yy,00} = -R^x{}_{0x0}.
\end{aligned}
\tag{61}
$$

So for example for two particles initially separated in the x direction:

$$
\frac{\partial^2}{\partial t^2}\xi^x = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xx}, \qquad \frac{\partial^2}{\partial t^2}\xi^y = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xy} \tag{62}
$$

Similarly for two particles separated in the y:

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\xi^y &= \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{yy} = -\frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xx}, \\
\frac{\partial^2}{\partial t^2}\xi^x &= \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h^{TT}_{xy}
\end{aligned}
\tag{63}
$$

Polarization

Let's assume a gravitational wave of the form:

$$
\begin{aligned}
ds^2 = -dt^2 + [1 - \epsilon h_{xx}(t-x)]dx^2 \\
+[1 + \epsilon h_{xx}(t-x)]dy^2 + dz^2.
\end{aligned}
\tag{64}
$$

The proper distance between two particles in the $(x-y)$ plane which have positions (x_0, y_0) and (x_0+dx, y_0) is given by:

$$
ds^2 = [1 - \epsilon h_{xx}(t-x)]dx^2 \tag{65}
$$

This is showing that the distance along the x axis is being stretched and squeezed along the x axis, while if we look at two particles which lay at the same value of y will be stretched and squeezed along the y axis. We can think of a ring of particles laying in a plane perpendicular to the z direction and the ring will be stretched and elongated in an ellipse whose major axis is switching back and forth between the y and x axis and vice versa for the minor one. This is called a $+$ polarized wave. What about the crossed term? We can pick a coordinate system:

$$
\begin{aligned}
ds^2 = -dt^2 + dx^2 + dy^2 \\
+[1 - \epsilon h_{xy}(t-x)]dxdy + dz^2.
\end{aligned}
\tag{66}
$$

We can perform a rotation through $45°$ in the $x-y$ plane

$$
x \to \bar{x} = \frac{1}{\sqrt{2}}(x+y), \qquad y \to \bar{y} = \frac{1}{\sqrt{2}}(x-y) \tag{67}
$$

$$ds^2 = -dt^2 + [1 + \epsilon h_{xy}(t-x)]d\bar{x}^2$$
$$+[1 - \epsilon h_{xy}(t-x)]d\bar{y}^2 + dz^2. \tag{68}$$

which can be seen that gives the same pattern just displaced $45°$. This is called a cross polarization.

An exact plane wave

Exact solutions of the Einstein's eqs. representing gravitational waves can be found. For this it is better to work in so called null coordinates:

$$u = t - z \tag{69}$$
$$v = t + z \tag{70}$$

The Minkowski metric in these coordinates will be:

$$ds^2 = -dudv + dx^2 + dy^2 \tag{71}$$

In looking for a solution representing a plane wave makes sense to look for:

$$ds^2 = -dudv + f^2(u)dx^2 + g^2(u)dy^2 \tag{72}$$

where f and g will need to be determined and we expect the solution to be a forward moving so we expect it to be only functions of u. The only non-vanishing connection coefficients are (dot means derivative respect to u);

$$\Gamma^x{}_{xu} = \dot{f}/f, \qquad \Gamma^y{}_{yu} = \dot{g}/g$$
$$\Gamma^v{}_{xx} = 2\dot{f}/f. \qquad \Gamma^v{}_{yy} = 2\dot{g}/g$$
$$R^x{}_{uxu} = -\ddot{f}/f, \qquad R^y{}_{uyu} = -\ddot{g}/g,$$

The only remaining vacuum field equation is:

$$\ddot{f}/f + \ddot{g}/g = 0 \tag{73}$$

These are actually a family of solutions. We can make an arbitrary pick for g and this would determine f. i.e. if we pick f such that

$$\ddot{f}/f = h(u) \tag{74}$$

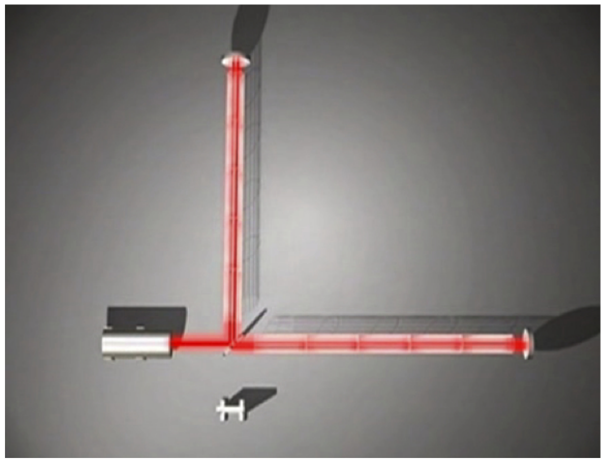
all we have to do is solve for

$$\ddot{g}/g = -h(u) \tag{75}$$

The solutions are determined up to two constants of integration. This family of solutions are called linearly polarized plane gravitational waves. They represent plane-fronted gravitational waves, far from the sources. It can be shown that they can cast in a form that is similar to first order to the general solutions we found in the weak field limit.

Measuring changes in distance with light

Gravitational wave detection started with the efforts of Joseph Weber who developed the technology of resonant bars. Resonant bars achieved very high sensitivity, particularly after the incorporation of cryogenic techniques to reduce thermal noise in the materials. Over the last twenty years interferometry has become the technology of choice to detect gravitational waves. The following is a simple sketch of an interferometer:


A Michelson interferometer

The following formula apply to the calculation of the travel time for a photon in the arms of an interferometer of length L. If we use the metric for the weak field:

$$\begin{aligned} ds^2 = -cdt^2 + (1 + h_{xx}(2\pi ft - \vec{k}\cdot\vec{x})dx^2 \\ +(1 - h_{xx}(2\pi ft - \vec{k}\cdot\vec{x})dy^2 + dz^2 \end{aligned} \tag{76}$$

we can think that we have a wave with a plus polarization traveling along the z direction: Then the proper travel time for a photon traveling round trip on the arm of an interferometer of length L is, using the fact that $ds = 0$:

$$\begin{aligned} \int_0^{\tau_{out}} dt = \frac{1}{c}\int_0^L \sqrt{1 + h_{xx}} \\ \approx \frac{1}{c}\int_0^L 1 + \frac{1}{2}h_{xx}(2\pi ft - \vec{k}\cdot\vec{x})dx \end{aligned} \tag{77}$$

and

$$\begin{aligned} \int_{\tau_{out}}^{\tau_{ret}} dt = -\frac{1}{c}\int_L^0 (1 + \frac{1}{2}h_{xx}(2\pi ft - \vec{k}\cdot\vec{x})dx \\ \tau = \frac{2L}{c} + \frac{1}{2c}\int_0^L 1 + \frac{1}{2}h_{xx}(2\pi ft - \vec{k}\cdot\vec{x})dx \\ -\frac{1}{2c}\int_L^0 (1 + \frac{1}{2}h_{xx}(2\pi ft - \vec{k}\cdot\vec{x})dx \end{aligned} \tag{78}$$

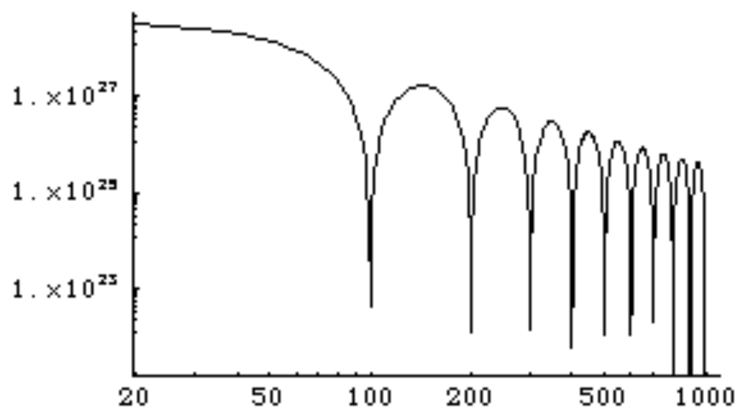
If $2\pi f_{gw} \ll 1$ the metric perturbation can be treated as constant during the time any given wavefront is in the detector. Let's assume $h(t) = h\exp(i2\pi f_{gw}t)$,

$$\int_0^{\tau_{out}} dt = \frac{L}{c} + \frac{h}{4\pi i f_{gw}}\left[e^{i2\pi f_{gw}L/c} - 1\right],$$
$$\int_{\tau_{out}}^{\tau_t} dt = \frac{L}{c} + \frac{h}{4\pi i f_{gw}} e^{i2\pi f_{gw}2L/c}\left[1 - e^{-i2\pi f_{gw}L/c}\right], \qquad (79)$$

Then the final $\delta\tau$ where $sinc(x) = sin(x)/x$ is:

$$\delta\tau = h\tau_t sinc(\pi f_{gw}\tau_{t0}) e^{i\pi f_{gw}\tau_{t0}}$$
$$\delta\phi(t) = h\tau_t \frac{2\pi c}{\lambda} sinc(\pi f_{gw}\tau_{t0}) e^{i\pi f_{gw}\tau_{t0}} \qquad (80)$$

The purpose of the above calculation was to estimate the sensitivity and response of an interferometer to the passage through it of a gravitational wave in the best possible conditions (i.e. perpendicular to the plane of the detector). This gives us the transfer function of an interferometer, i.e. the frequency sensitivity for the detector:



Transfer function for an interferometer

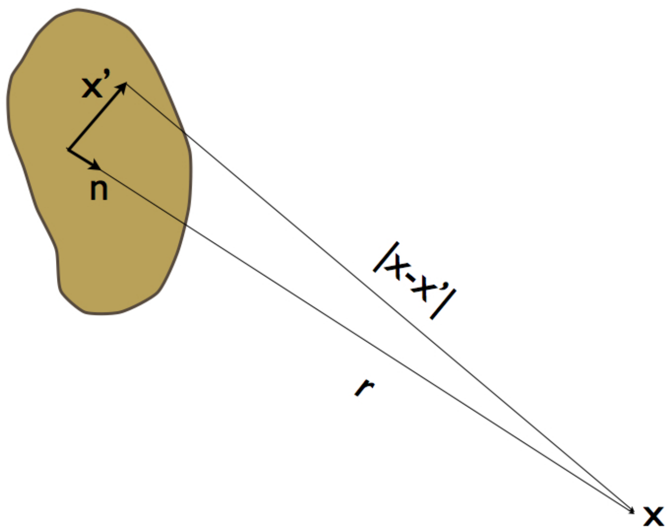
The graph above shows the magnitude of $\delta\phi$ as a function of the gravitational wave frequency (in Hz in the horizontal axis. The arms are at $1,500km$ distance.The light wavelength is $\lambda = 0.5\mu m$. This a LogLogPlot in Mathematica. The y axis is $\delta\phi$Æ$/h$.

Slow motion wave generation

We will see that the lowest possible order of radiation originates in the temporal variation of mass distribution at the quadrupole level. We will assume that the field point $\vec{x}$ is in the radiation zone, far from the source.

$$\Box \bar{h}^{\mu\nu} = -\kappa T^{\mu\nu} \tag{81}$$

We will assume that the field point $\vec{x}$ is in the radiation zone, far from the source.



The field far from the source

The so called "retarded" solution (like in E&M) is:

$$\bar{h}^{\mu\nu} = -\frac{\kappa}{4\pi} \int \frac{T^{\mu\nu}(t - |\vec{x} - \vec{x'}|, \vec{x'})}{|\vec{x} - \vec{x'}|} d^3x' \tag{82}$$

You can compare with the analogous E&M treatment in books like Jackson's using that $\nabla^2(1/|\vec{x} - \vec{x'}|) = -4\pi\delta(\vec{x} - \vec{x'})$ We assume that the field point $\vec{x}$ is in a region far from the source. So we can replace $|\vec{x} - \vec{x'}|$ in the denominator by $|\vec{x'}|$ in (57). We will assume that the time dependence is not very strong, and then replace $t - |\vec{x} - \vec{x'}|$ by $t - |\vec{x'}|$, and change our notation such that $r \equiv |\vec{x'}|$ and get:

$$\bar{h}^{\mu\nu} = -\frac{\kappa}{4\pi r} \int T^{\mu\nu}(t - r, \vec{x'}) d^3x' \tag{83}$$

Note: The condition for both assumptions above could be summarized: $r \ll \lambda$ and all the equivalent conditions: $fb/c \ll 1$ or $b\lambda \ll 1$, or $v/c \ll 1$, where f and λ are the frequency and wavelength of the radiation respectively. b is the typical dimension of the system. The Bianchi conditions obligate $T^{\mu\nu}$ to satisfy:

$$T^{\mu\nu}{}_{,\nu} = 0 \tag{84}$$

It can be written:

$$T^{k0}{}_{,t} = -\partial_l T^{kl} \qquad\qquad T^{00}{}_{,l} = -\partial_l T^{0l} \tag{85}$$

Using the first of equations (60) we can see that if we integrate by parts after taking a volume integral the following identity can be derived:

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int (T^{k0}x^l + T^{l0}x^k)d^3x = \\ -\frac{1}{2}\int (\partial_l T^{kl}x^l + \partial_l T^{0l}x^k)d^3x = \int T^{kl}d^3x\end{aligned} \tag{86}$$

Working the same manner the second equation in (60):

$$\int (T^{k0}x^l + T^{l0}x^k)d^3x = \frac{\partial}{\partial t}\int T^{00}x^k x^l d^3x \tag{87}$$

Combining (61) and (62) we find:

$$\int T^{kl}d^3x = \frac{1}{2}\frac{\partial^2}{\partial t^2}\int T^{00}x^k x^l d^3x \tag{88}$$

For slow motions of matter then we get:

$$T^{00} \approx \rho \tag{89}$$

and we obtain applying our results to (58):

$$\bar{h}^{kl}(t,\vec{x}) = -\left[\frac{\kappa}{8\pi r}\frac{\partial^2}{\partial t^2}\int \rho(\vec{x}')x'^k x'^l d^3x'\right]_{t-r} \tag{90}$$

This integral can be written in terms of the quadrupole momentum tensor:

$$Q^{kl} = \int (3x'^k x'^l - r'^2\delta_k^l)\rho(\vec{x'})d^3x' \tag{91}$$

This quadrupole momentum appears naturally when one performs a decomposition of the newtonian potential solution. ie. The well known solution is:

$$\Phi(\vec{x}) = -\int \frac{G\rho(\vec{x'})}{|\vec{x}-\vec{x'}|}d^3x' \tag{92}$$

When the point $\vec{x}$ is outside the region of the mass distribution, a multipole expansion can be done for the potential. The solution after performing the integrals is:

$$\Phi(\vec{x}) = -\frac{GM}{r} - \frac{G}{r^3}\sum_k x^k D^k - \frac{G}{2}\sum_{k,l} Q^{kl}\frac{x^k x^l}{r^5} + \cdots \tag{93}$$

where:

$$M = \int \rho(\vec{x'})d^3x' \tag{94}$$

$$D^k = \int x'^k \rho(\vec{x'})d^3x' \tag{95}$$

$$Q^{kl} = \int (3x'^k x'^l - r'^2\delta_k^l)\rho(\vec{x'})d^3x' \tag{96}$$

Using (71) in (65) we get:

$$\bar{h}^{kl}(t,\vec{x}) = -\frac{\kappa}{8\pi r}\frac{1}{3}\left[\frac{\partial^2}{\partial t^2}Q^{kl} + \delta_k^l\frac{\partial^2}{\partial t^2}\int r'^2\rho(\vec{x}')d^3x'\right]_{t-r} \tag{97}$$

But the δ_k^l term can never give a zero trace term according to what we learned about plane wave solutions. The final solution is:

$$\bar{h}^{kl}(t,\vec{x}) = -\frac{\kappa}{8\pi r}\frac{1}{3}\ddot{Q}^{kl} \tag{98}$$

The monopole contribution to radiation does not appear due to the existence of only one type of mass and the conservation of it when we include the rest energy. The dipole does not appear either due to the conservation of momentum. So we found that the origin of gravitational radiation to lowest order is in the acceleration of the quadrupole momentum. Notice that the quadrupole tensor of a spherical mass distribution is zero. I state for completion but without proof the energy radiated by a gravitational source:

$$-\frac{dE}{dt} = \frac{G}{45c^5}\dddot{Q}^{kl}\dddot{Q}^{kl} \tag{99}$$