# Lesson 1 - 2025
# Cosmology
# or The Natural History of Everything

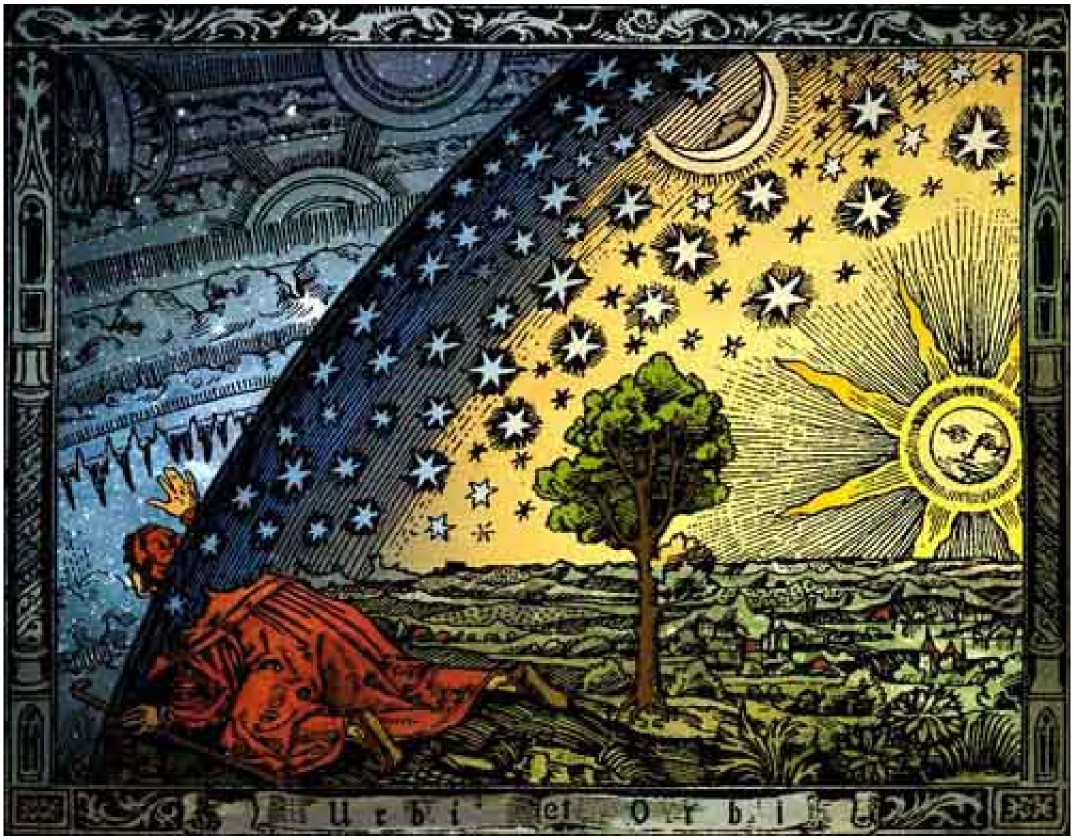
Mario Diaz

March 26, 2026

**What is Cosmology?**

*most of the material follows d'Inverno's "Introducing Einstein's Relativity", although I kept the signature we have used throughout the course and not the one the author utilizes in his book.*
I follow Schutz in discussing luminosity distance and the expansion of the universe. Similarly when I introduce dark energy. The layout for this part of the course will follow the discussion of the following issues or questions.

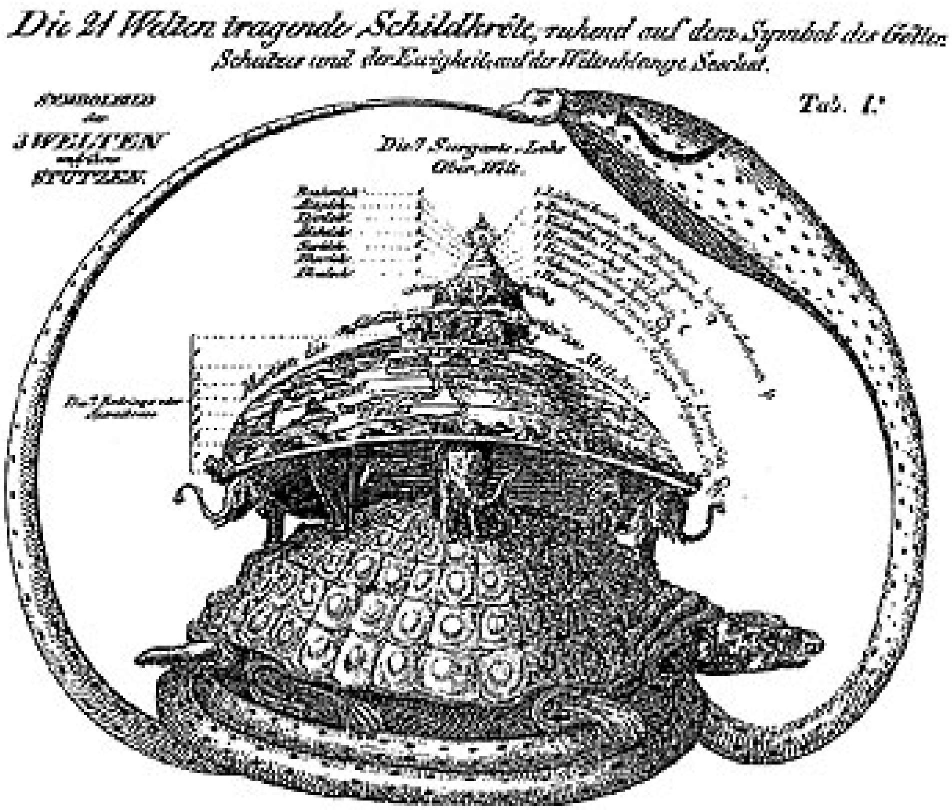
- How does everything fit together?
- Olbers paradox
- Newtonian Cosmology
- The Cosmological Principle
- Weyl's postulate
- Relativistic Cosmology



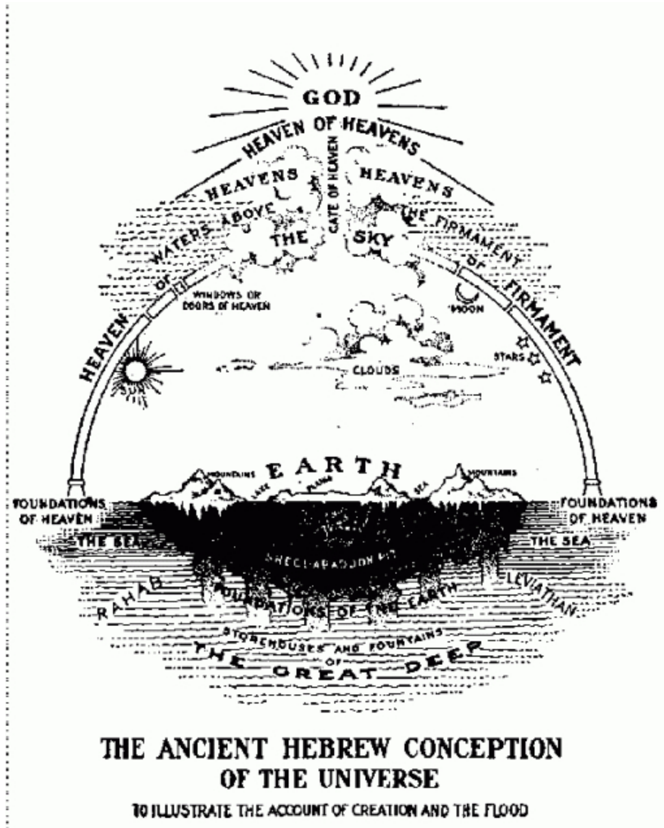
The ancient world of the Westerners


The Cosmos according to the Incas


An according to the Mayans




The world for the Ancient Hindus




The world for the Ancient Hebrew

**Olbers' paradox**

In 1826 Olbers postulated his famous paradox:

- How come the sky is so dark if it's filled with stars in an infinite universe?

It is not difficult to see that if we look at the center of a shell of radius $r$ with a total luminosity $l$ provided by the average luminosity of the stars contained multiplied by the number of stars inside this volume the intensity of the light produced at the center of the shell will be this total luminosity divided by the area of it, i.e.:

$$\frac{(4\pi r^2 dr)l}{4\pi r^2} = ldr \tag{1}$$

We clearly get the total intensity at P by integrating over all the shells around P up to infinity:

$$\int_0^\infty ldr = \infty! \tag{2}$$

But the sky is dark at night! The paradox is that a static, infinitely old universe with an infinite number of stars distributed in an infinitely large space would be bright rather than dark. We could have taken into consideration: Absorption of light by stars in the line of sight. Olbers postulated the existence of a tenuous gas which would absorb the radiation (this is an inconsistent argument from the point of view of Thermodynamics). The expansion of the Universe would definitely be able to provide an explanation for the paradox. For a more complete explanation of the paradox and interesting alternative resolutions see: $http://en.wikipedia.org/wiki/Olbers'\_paradox$ (excellent account, including historical precedents).

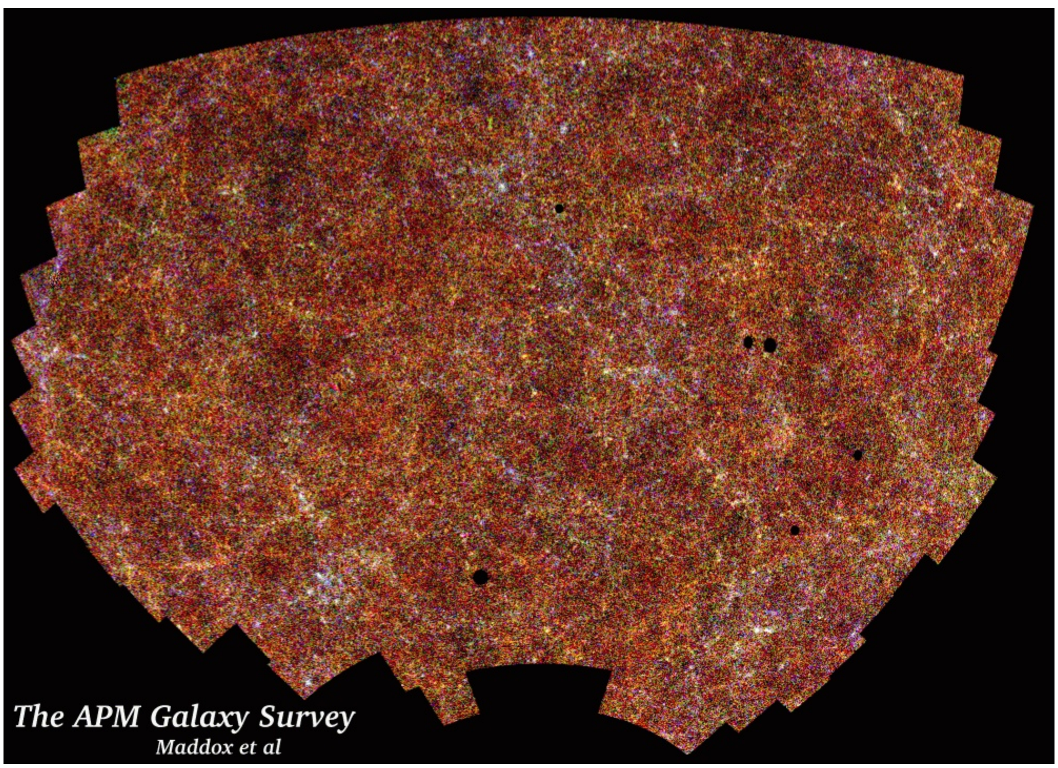
**The Homogeneous Isotropic Cosmological Model**

At any given instant of time:

- **Homogeneity:** At different space points all observable quantities are the same. This means no matter which region we look at they all look similar at a given scale. We find the same proportion of spiral galaxies, elliptical galaxies, irregular galaxies. The clustering of galaxies at larger scales is observed to be distributed equally along the universe. No preferred point or location in space.
- **Isotropy:** At any space point all directions are equivalent. No special vantage point. No rotation (that would indicate an axis and then a preferred direction).

We could investigate:

- Relativistic models (i.e. using the general theory of relativity).
- Newtonian gravity.




Galaxy Survey 30° across with a million galaxies up to a distance of 2 billion light years.

**Newtonian Cosmology**

- A spherically symmetric distribution of matter does not give rise to a gravitational force inside a spherical cavity concentric to the matter distribution.
- We can assume a distribution of velocities for the particles that make up this fluid: $v = Hr$

Edwin Hubble discovered his law when studying distance to galaxies and their spectra redshift in 1929.

*Features of our model:*

- The matter at the origin of the coordinate system is at rest.
- Only the assumed type of velocity distribution is isotropic and homogeneous.
- An observer moving along with the particles sees all neighboring particles receding.

*Evolution of the model:*

$$v = Hr \qquad\qquad \frac{dr}{dt} = Hr \tag{3}$$

$$r_{AB}(t) = r_{AB}(t_0)e^{\int_{t_0}^{t} H(t)dt} \tag{4}$$

*Density evolution*

Let's assume a mass $M$,volume of radius $R$, then:

$$\rho = \frac{M}{4/3\pi r^3} \qquad \rightarrow \qquad \frac{d\rho}{dt} = \frac{-3M}{4/3\pi r^4}\frac{dr}{dt} \tag{5}$$

And if we substitute:

$$dr/dt = \nu = Hr \tag{6}$$

$$\frac{d\rho}{dt} = \frac{-3M}{4/3\pi r^4}Hr = -3\rho H \tag{7}$$

And we have:

$$\frac{d\rho}{dt} = -3\rho H \tag{8}$$

*How does the velocity change?*
We now will use Newton's gravity:

$$\frac{dv}{dt} = a = -\frac{GM}{r^2} = -\frac{G\frac{4}{3}\pi\rho r^3}{r^2} = -\frac{4}{3}\pi G\rho r \tag{9}$$

How do $H$ and $\rho$ evolve?
Using (6), $\rho = \frac{M}{4/3\pi r^3}$ and $a = \frac{d^2r}{dt^2}$ we get:

$$\frac{d^2r}{dt^2} = \frac{d}{dt}(Hr) = \frac{dH}{dt}r + H\frac{dr}{dt} = r\frac{dH}{dt} + HHr \tag{10}$$

From where we have:

$$\frac{dH}{dt} = -H^2 - \frac{4}{3}\pi G\rho \tag{11}$$

$$\frac{d\rho}{dt} = -3\rho H \tag{12}$$

Eqs (11) and (12) form a complete system of equations.
We can now multiply (9) by $\frac{dr}{dt}$

$$\frac{d^2r}{dt^2}\frac{dr}{dt} = -\frac{GM}{r^2}\frac{dr}{dt} \tag{13}$$

which yields:

$$\frac{1}{2}\frac{d}{dt}(\dot{r}^2) = GM\frac{d}{dt}\left(\frac{1}{r}\right) \tag{14}$$

$$\frac{d}{dt}\left[\frac{1}{2}(\dot{r}^2) - \frac{GM}{r}\right] = 0 \tag{15}$$

$$\frac{1}{2}(\dot{r}^2) - \frac{GM}{r} = constant \tag{16}$$

We can calculate the constant at $t = t_0$

$$A = \frac{1}{2}(H_0^2 R_0^2) - \frac{4}{3}\pi G \rho_0 R_0^2 \tag{17}$$

and using

$$M = \frac{4}{3}\pi \rho_0 R_0^3 \tag{18}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{8}{3}\pi G \rho_0 \frac{R_0^3}{r} - \frac{8}{3}\pi G R_0^2 \left(\rho_0 - \frac{3H_0^2}{8\pi G}\right) \tag{19}$$

where we can define a critical density:

$$\rho_c = \frac{3H_0^2}{8\pi G} \tag{20}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{8}{3}\pi G \rho_0 \frac{R_0^3}{r} - \frac{8}{3}\pi G R_0^2 \left(\rho_0 - \rho_c\right) \tag{21}$$

Now we can do a qualitative analysis:

- $dr/dt > 0$
- r increases with time

Then in the past $\frac{8}{3}\pi G \rho_0 \frac{R_0^3}{r}$ was larger and also $\frac{dr}{dt}$ was large. So in the past should have been a time when $\rightarrow$

- $r = 0$
- $\frac{dr}{dt} = +\infty$

This is the **Big Bang!**

But the future depends on $(\rho - \rho_c)$.

We can write (21) defining two arbitrary constants:

$$\left(\frac{dr}{dt}\right)^2 = \frac{B}{r} - C\left(\rho_0 - \rho_c\right) \tag{22}$$

As $R$ grows: If $\rho_0 > \rho_c$
and $r$ is very small but $1/r$ grows until

$$r = \frac{B}{C(\rho_0 - \rho_0)} \tag{23}$$

And then $dr/dt = 0$ and the expansion stops!. But if $\rho_0 < \rho_c$
$dr/dt > 0$ and the expansion continues forever!

$$\frac{dr}{dt} = [C(\rho_c - \rho_0)]^{\frac{1}{2}} \tag{24}$$

And then if $\rho_0 = \rho_c$

$$\left(\frac{dr}{dt}\right)^2 = \frac{B}{r} \qquad \rightarrow \qquad r(t) = Dt^{2/3} \tag{25}$$

**Relativistic Cosmology**
Three postulates are the basis of RC:

- the cosmological principle: on large scale the universe looks the same to any observer (the universal Copernican Principle).
- Weyl's postulate: the universe can be represented by a perfect fluid, where the particles of the fluid are the galaxies.
- general relativity

Weyl's postulate can be expressed mathematically saying that there is a time, i.e. the proper time co-moving with the galaxies. i.e. the galaxies move on time-like geodesics defining orthogonal hypersurfaces of constant coordinates. This orthogonality can be expressed:

$$ds^2 = dt^2 - h_{ij}dx^i dx^j \tag{26}$$

$t$ is the cosmic time. The world map is the series of events on the surfaces of simultaneity (same t). The world picture is the set of events an observer sees in her past light cone at a given cosmic time. Due to the fact that we require isotropy and homogeneity we need to require that the spatial part of the metric be conformal in time, i.e. that the metric is multiply by an overall factor depending of time:

$$h_{ij} = S^2(t)g_{ij}(x^k)dx^i dx^j \tag{27}$$

The ratio of two values of S at different times is the magnification factor (scale factor). We will also require that the curvature at each point be constant, i.e. given a time slice the curvature of the surface has to be constant otherwise the isotropy and homogeneity will be lost. It can be shown that spaces of constant curvature are defined:

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{28}$$

where K is the constant curvature. Since the 3-space is isotropic about every point, it must be spherically symmetric. We can use the 3-space metric defined from (21) and (22) in Lesson 11:

$$d\sigma^2 = g_{ij}dx^i dx^j = e^{\lambda}dr^2 + r^2 d\Omega^2, \tag{29}$$

where $\lambda = \lambda(r)$. The non-vanishing components of the RIcci tensor are:

$$R_{11} = \lambda'/r, R_{22} = cosec^2\theta R_{33}, \tag{30}$$

$$R_{33} = 1 + \frac{1}{2}re^{-\lambda}\lambda' - e^{-\lambda}. \tag{31}$$

Condition (28) yields:

$$\lambda'/r = 2Ke^{\lambda}, 1 + \frac{1}{2}re^{-\lambda}\lambda' - e^{-\lambda} = 2Kr^2. \tag{32}$$

The solutions is:

$$e^{-\lambda} = 1 - Kr^2. \tag{33}$$

This gives us the metric for the 3-space of constant curvature:

$$d\sigma^2 = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2, \tag{34}$$

It is more convenient to define:

$$r = \frac{\bar{r}}{(1 + \frac{1}{4}K\bar{r}^2)}, \tag{35}$$

and the metric becomes:

$$d\sigma^2 = (1 + \frac{1}{4}K\bar{r}^2)^{-2}\left[d\bar{r}^2 + \bar{r}^2 d\Omega^2\right], \tag{36}$$

Combining with (27):

$$ds^2 = -dt^2 + S^2(t)\frac{d\bar{r}^2 + \bar{r}^2 d\Omega^2}{(1 + \frac{1}{4}K\bar{r}^2)^2}, \tag{37}$$

And one more effort: it is convenient to leave only the sign of the K (which is a scale factor) as a physically relevant parameter: If $K \neq 0$ we can define $k = K/\|K\|$. If we define also $r^* = \|K\|^{1/2} r$ we would get:

$$ds^2 = -dt^2 + \frac{S^2(t)}{|K|}\left(\frac{dr^{*2}}{1 - r^{*2}} + r^{*2}(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{38}$$

and defining a rescaled scaled function as well:

$$R(t) = S(t)/\|K\|^{1/2} \qquad if \qquad K \neq 0, \tag{39}$$

$$R(t) = S(t) \qquad if \qquad K = 0 \tag{40}$$

we get after dropping the stars, finally!:

$$ds^2 = -dt^2 + R^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right) \tag{41}$$

or in the $\bar{r}$ coordinate:

$$ds^2 = -dt^2 + R^2(t)\frac{d\bar{r}^2 + \bar{r}^2 d\Omega^2}{(1 + \frac{1}{4}K\bar{r}^2)^2}, \tag{42}$$

where $k = +1, -1, 0$.

(41) is called the Robertson-Walker metric.

*The associated geometries*

$k = +1$

We see in (41) that the coefficient of $dr^2$ becomes singular as $r \to 1$. We can go around with:

$$r = \sin\chi, \tag{43}$$

and,

$$dr = \cos\chi d\chi = (1 - r^2)^{1/2} d\chi, \tag{44}$$

and the 3-d part becomes:

$$d\sigma^2 = R_0^2 \left(d\chi^2 + \sin^2\chi d\Omega^2\right), \tag{45}$$

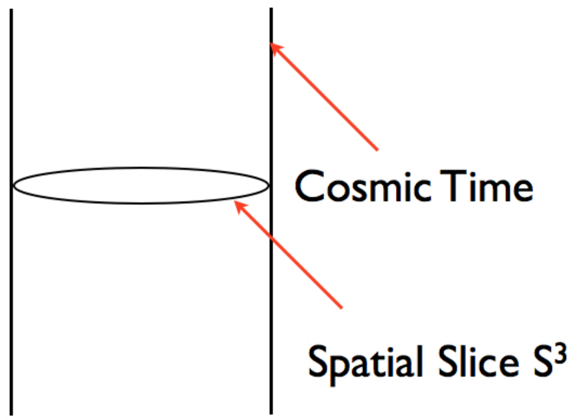
But now we can embed this 3-surface in a 4-dimensional Euclidean space $(w, x, y, z)$ where

$$\left.\begin{array}{r}w = R_0 \cos\chi, \\ x = R_0 \sin\chi \sin\theta \cos\phi, \\ y = R_0 \sin\chi \sin\theta \sin\phi, \\ z = R_0 \sin\chi \cos\theta.\end{array}\right\} \tag{46}$$

Now trivially:

$$d\sigma^2 = dw^2 + dx^2 + dy^2 + dz^2 = R_0^2 \left(d\chi^2 + \sin^2\chi d\Omega^2\right), \tag{47}$$

which is in agreement with (44).




The topology with k=+1

$k = 0$ If we look at (41) at a given time $t = t_0$ the spatial part of the metric can become with the following coordinate choice:

$$\left.\begin{aligned} x &= R_0 \sin\theta\cos\phi, \\ y &= R_0 \sin\theta\sin\phi, \\ z &= R_0 \cos\theta. \end{aligned}\right\} \tag{48}$$

Then the metric becomes

$$d\sigma^2 = dx^2 + dy^2 + dz^2, \tag{49}$$

And the topology is the same as the $k = +1$ case.

$k = -1$

We can introduce a new coordinate $r = \sinh\chi$ and then,

$$dr = \cosh\chi d\chi = (1 + r^2)^{1/2} dr, \tag{50}$$

so

$$d\sigma^2 = R_0^2 \left( d\chi^2 + \sinh^2\chi d\Omega^2 \right), \tag{51}$$

Notice that now we can embed this 3-surface in a a flat Minkowski space:

$$d\sigma^2 = -dw^2 + dx^2 + dy^2 + dz^2, \tag{52}$$

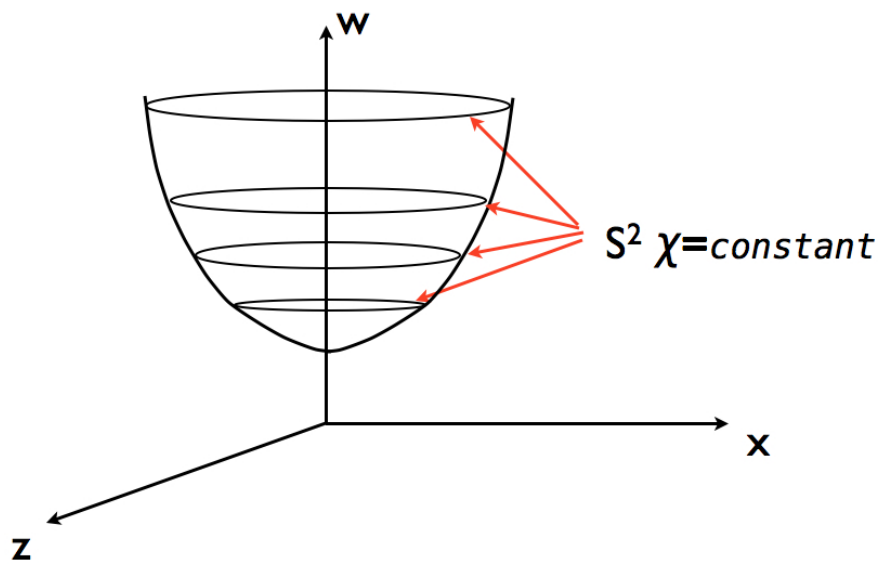
with the coordinates defined:

$$\left.\begin{aligned} w &= R_0 \cosh\chi, \\ x &= R_0 \sinh\chi \sin\theta \cos\phi, \\ y &= R_0 \sinh\chi \sin\theta \sin\phi, \\ z &= R_0 \sinh\chi \cos\theta. \end{aligned}\right\} \tag{53}$$

The equations imply that:

$$-w^2 + x^2 + y^2 + z^2 = R_0^2 \tag{54}$$

so the 3-surface is a three dimensional hyperboloid in four dimensional Minkowkski space.




The topology with k=-1

The 2-surfaces in the picture are 2-spheres of surface area:

$$A_\chi = 4\pi R_0^2 \sinh^2\chi$$

$\chi$ ranges from $0 \to \infty$ The 3-volume can become infinite.

**Friedmann's equation**
We will now work to develop relativistic cosmological models. We need:

- the FRW metric

$$ds^2 = -dt^2 + R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{55}$$

- Weyl's postulate

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu} \tag{56}$$

- Einstein's cosmological eqs:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{57}$$

Then using that in our comoving coordinate system $\vec{u} = (1,0,0,0)$, the field equations become,

$$3\frac{\dot{R}^2 + k}{R^2} - \Lambda = 8\pi\rho, \tag{58}$$

$$\frac{2R\ddot{R} + \dot{R}^2 + k}{R^2} - \Lambda = -8\pi p, \tag{59}$$

Due to considerations of isotropy and homogeneity $\rho$ and $\pi$ can be only functions of time.
Differentiating (57) respect to time, multiply by $1/8\pi$ and add the result to (58) multiplied by $-3\dot{R}/8\pi R$ we get:

$$\dot{\rho} + 3p\frac{\dot{R}}{R} = -\frac{3}{8\pi}\frac{\dot{R}}{R}\left(\frac{3\dot{R}^2}{R^2} + \frac{3k}{R^2} - \Lambda\right) = -3\rho\frac{\dot{R}}{R}, \tag{60}$$

Multiplying by $R^3$ we can rewrite this:

$$\frac{d}{dt}\left(\rho R^3\right) + p\frac{d}{dt}\left(R^3\right) = 0, \tag{61}$$

But $R^3(t)$ is the volume of the fluid we are considering $V$. And $\rho V$ the total mass-energy in the volume $V$. But then we can rewrite (60),

$$dE + pdV = 0 \tag{62}$$

Notice that this is the law of conservation of energy. This is the result of satisfying the Bianchi identities. But if we consider experimental evidence, $p/\rho < 10^{-5}$, so we can assume $p = 0$. In that case (58) integrates immediately. First we need to multiply it by $\dot{R}$ throughout, and then identify that the left hand side is a total time derivative of the expression

$$R(\dot{R} + k) - \frac{1}{3}\Lambda R^3 = C \tag{63}$$

with $C$ a constant of integration, which can be quickly identified using (57) as

$$C = \frac{8}{3}\pi R^3 \rho \tag{64}$$

This is twice the mass content of a spherical volume of a Euclidean universe of radius $R$ and density $\rho$. But we can use (63) now to eliminate $\rho$ in (57):

$$\dot{R}^2 = \frac{C}{R} + \frac{1}{3}\Lambda R^2 - k \tag{65}$$

I will call R, which some authors call the scale factor, the "radius of the Universe". (64) is called the **Friedmann's equation**. Compare (64) with (21)! The only difference is the cosmological constant term. Notice that in the Newtonian derivation we were looking at an expanding fluid with zero pressure.

**Propagation of light**

An observer sees light from a galaxy (which is receding). A radial null geodesic will have:

$$ds^2 = d\theta = d\phi = 0 \tag{66}$$

So using (41) we get

$$\frac{dt}{R(t)} = \pm \frac{dr}{(1-kr^2)^{\frac{1}{2}}} \tag{67}$$

We can think of a time $t_0$ at which the observer is receiving light from a galaxy situated at a point with $r = r_1$ and emitted at time $t_1$. Then

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = -\int_{r_1}^{0} \frac{dr}{(1-kr^2)^{\frac{1}{2}}} = f(r_1), \tag{68}$$

where,

$$f(r_1) = \begin{cases} \sin^{-1} r_1 & \text{if } k = +1, \\ r_1 & \text{if } k = 0, \\ \sinh^{-1} r_1 & \text{if } k = -1. \end{cases}$$

Let's consider now two successive rays of light which are emitted by the galaxy at times $t_1$ and $t_1 + dt$ which are received by our observer at times $t_0$ and $t_0 + dt_0$. Following (67) we see that even if we calculate the left hand side for different intervals of time it will still be a function only of the position of the galaxy respect to the observer (i.e. $r_1$):

$$\int_{t_1+dt_1}^{t_0+dt_0} \frac{dt}{R(t)} = \int_{t_1}^{t_0} \frac{dt}{R(t)} = f(r_1), \tag{69}$$

and then:

$$\int_{t_1+dt_1}^{t_0+dt_0} \frac{dt}{R(t)} - \int_{t_1}^{t_0} \frac{dt}{R(t)} = 0 =, \tag{70}$$

$$\int_{t_0}^{t_0+dt_0} \frac{dt}{R(t)} - \int_{t_1}^{t_1+dt_1} \frac{dt}{R(t)} \tag{71}$$

If $R(t)$ doesn't vary much over $dt_1$ and $dt_0$ we get:

$$\frac{dt_0}{R(t_0)} = \frac{dt_1}{R(t_1)}. \tag{72}$$

The galaxies move along world lines on which the coordinates $r, \theta, \phi$ are constant.
Consequently, $ds^2 = dt^2$ which implies that $t$ measures the proper time along the fluid particles (galaxies) world lines. $dt_1$ and $dt_0$ are the proper time intervals between the two light rays emitted by the galaxy and perceived by the observer. This means according to (71) that the time interval measured by the observer is $R(t_0)/R(t_1)$ times the time interval measured by someone at the emitter galaxy. The universe is expanding, so $t_0 > t_1$ which implies $R(t_0) > R(t_1)$. This implies that the observer $O$ will perceive a redshift $z$ given by

$$1 + z = \frac{\nu_1}{\nu_0} = \frac{R(t_0)}{R(t_1)}. \tag{73}$$

where $\nu_1$ and $\nu_0$ are the frequencies measured by the emitter and the receiver. This the cosmological redshift.
For short time differences $t_0 = t_1 + dt$ and so (72) will become:

$$1 + z = \frac{\nu_1}{\nu_0} = \frac{R(t_0)}{R(t_0 - dt)} \simeq \frac{R(t_0)}{R(t_0) - \dot{R}_{t_0} dt} \tag{74}$$

$$\approx 1 + \frac{\dot{R}(t_0)}{R(t_0)} dt \tag{75}$$

If we integrate,

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_{t_1}^{t_1+dt} \frac{dt}{R(t)} \approx \frac{dt}{R(t_1)} = \tag{76}$$

$$\frac{dt}{R(t_0 - dt)} \approx \frac{dt}{R(t_0)} \tag{77}$$

In the case of small $r$, we get from $f(r)$ as defined in (67) and the formula below,

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = f(r_1) \approx r_1, \tag{78}$$

So we can write now:

$$\frac{dt}{R(t_0)} \approx r_1 \tag{79}$$

And from (74)

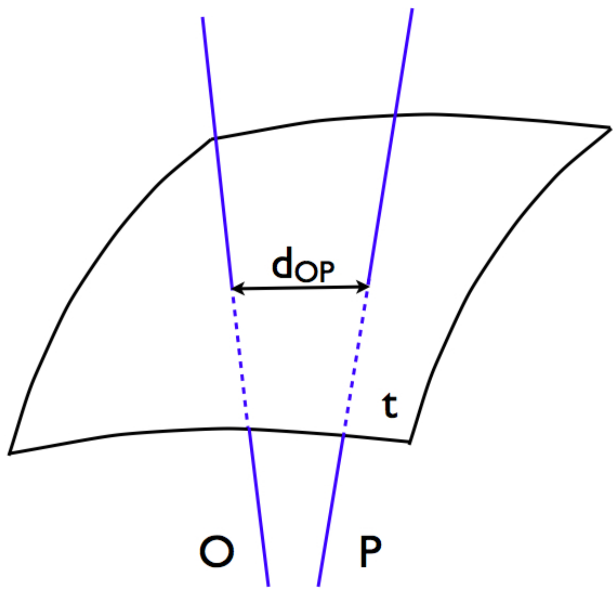
$$z \approx \dot{R}(t_0) r_1 \tag{80}$$

**Distance in Cosmology**

An easy way to measure distance is to use the world time and measure the distance (absolute at the same time on a given slice) between particles by just measuring the proper distance along a geodesic line.
We set $dt = d\theta = d\phi = 0$ in (41). Then,

$$d_A = R(t) \int_0^{r_1} \frac{dr}{(1 - kr^2)^{1/2}} \tag{81}$$

But we would need to know $R_1$.
Because what we know it is the apparent luminosity of the particle, i.e. the galaxy or nebula, we may try it. If $E$ is the total luminous energy radiated per unit of time by the galaxy, and $I$ the intensity of the radiation measured by the observer per unit of area and time.




Cosmological distance

We can define the distance as $(E/4\pi I)^{1/2}$. There is an issue with the time though. We need to account for the Doppler shift of the light (a result of the expanding universe!) so the number of photons will be reduced but also their energy (frequency) will be reduced. So the redshift factor enters twice! and we have:

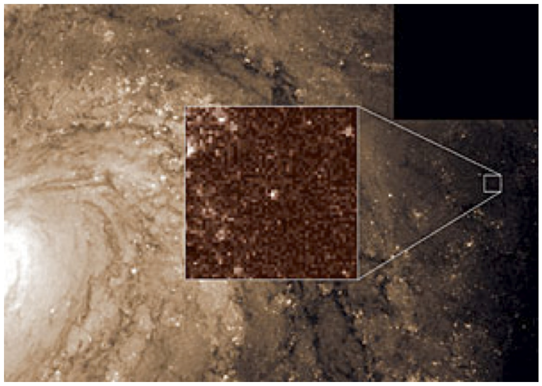
$$d_L^2 = \frac{E}{4\pi I(1+z)^2} \tag{82}$$

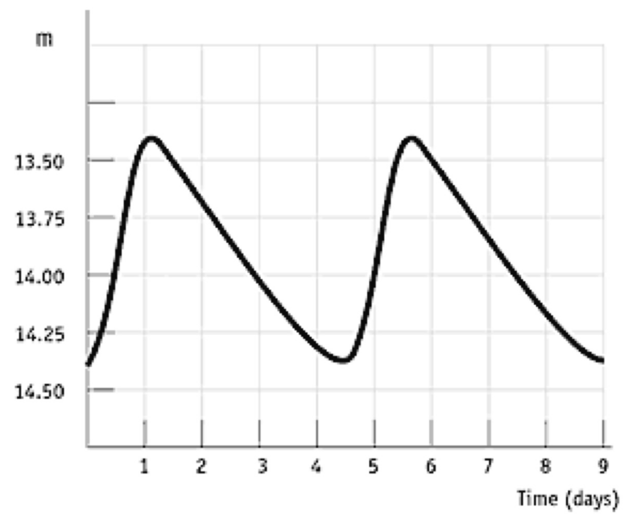
This is called the luminosity distance. This formula is just an approximation. The main problem is that we can not measure in general $E$.

**A digression on measuring distance to the stars**

In 1912 Henrietta Leavitt found the following relationship between absolute magnitude and the period of variable stars called Cepheids:

$$M = -2.78 \log P - 1.35 \tag{83}$$


Cepheid stars in M100




Light curve for a cepheid.

The apparent magnitude is related to the absolute magnitude (or energy emitted) and can be related to the energy received (which is itself related to the distance to the source by (74).
The relationship between the apparent magnitude and the energy received is given by:

$$m = constant - 0.4 \log_{10} E_R. \tag{84}$$

In a similar manner distances can be inferred from luminosity measurements of nearby galaxies,

$$L = 4\pi d^2 F$$

where $d$ is the distance to the galaxy and $F$ is the flux we measure. From this we can define the luminosity distance

$$d_L = \left(\frac{L}{4\pi F}\right)^{1/2}$$

What is the relationship between the luminosity distance and the cosmological scales? Suppose that our source gives only one type of photons of frequency $\nu_e$ at time $t_e$ What is the flux at a later time $t_0$? In an interval $\delta t_e$ the source emits:

$$N = L\delta t_e / h\nu_e$$

To find the flux we need to know the area of the sphere that these photons occupy at the time we observe them.

Integrating over the spherical angles using (54) we get:

$$A = 4\pi {R_0}^2 r^2$$

but the photons have redshifted by $1 + z = R_0/R(t_e)$ to a frequency $\nu_0$:

$$h\nu_0 = h\nu_e/(1 + z)$$

They would arrive during a time $\delta t_0$

$$\delta t_0 = \delta t_e (1 + z)$$

The flux of light at time $t_0$ is then $Nh\nu_0/(A\delta t_0)$, and then,

$$F = L/A(1 + z)^2$$

Then the luminosity distance $d_L$ is,

$$d_L = R_0 r(1 + z)$$

To get the final formula we need to have the value of $r$ as a function of the redshift $z$. All we need to do is to set $ds = 0$ in (54) which gives us

$$\frac{dr}{(1 - kr^2)^{1/2}} = -\frac{dt}{R(t)}$$

**Hubble's law**

If light coming from $P_1$ at time $t_1$ is observed "now" by an observer at $O$ at time $t_0$ where $t_1 < t_0$ the light will be extended over a sphere with centre at $P_0$ where $t = t_0$ and $r = r_1$ and passing through $O_0$ with $t = t_0$ and $r = 0$. The surface area of the sphere centered at $O_0$ is the same as the one centered at $P_0$. We have to remember that the 3-sphere is homogeneous. From (41) the line element of the 3-sphere is:

$$ds^2 = [R(t_0)r_1]^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{85}$$

The integration of $d\Omega^2$ will give the sphere with surface $4\pi R^2(t_0)r_1^2$ and consequently the observed intensity for the galaxy's light emitted at $P_1$ is

$$I = \frac{E}{4\pi r_1^2 R^2(t_0)(1 + z)^2}, \tag{86}$$

Comparing with (81) we get:

$$d_L = r_1 R(t_0). \tag{87}$$

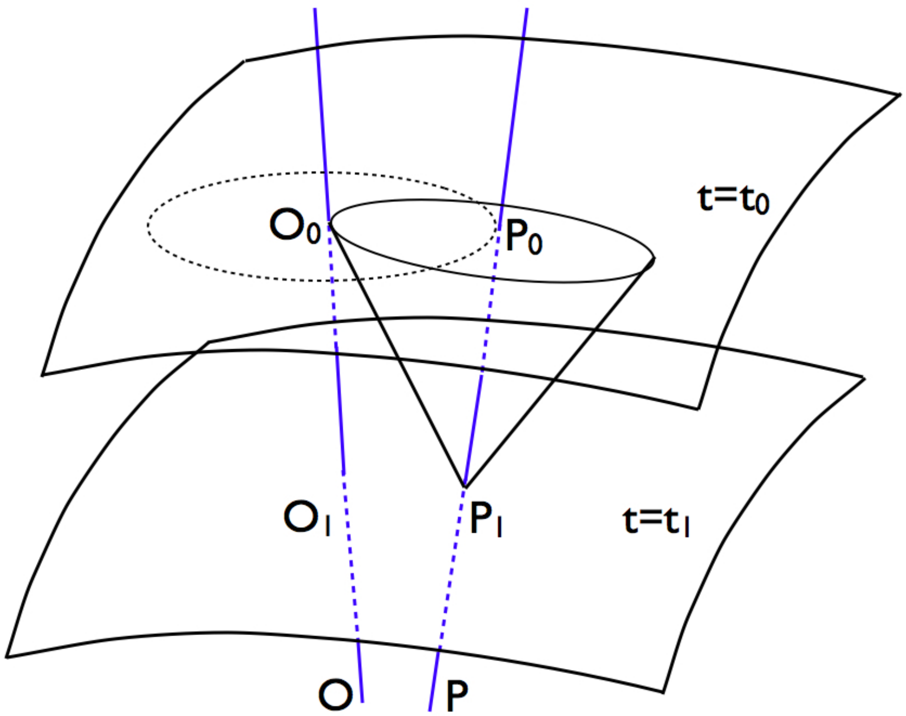
If we define the Hubble parameter

$$H(t) = \frac{\dot{R}(t)}{R(t)} \tag{88}$$

the we have

$$z \approx H(t_0)d_L, \tag{89}$$

$H_0$ is the value of the Hubble parameter at the current epoch an is called the Hubble constant.




The sphere of light from $P_1$ at $O_0$

According to the WMAP results the most current value of the Hubble's constant is $73.5 \pm 3.2$ km/sec/Mpc. This is $H_0 = 2.3 \times 10^{-18}$ 1/sec.
The Hubble time is $T = 1/H_0 = 4.35 \times 10^{17}$ sec. The velocity of recession of galaxies as measured by their redshift is proportional to its distance. The deceleration parameter $q$ is

$$q(t) = -\frac{R\ddot{R}}{\dot{R}^2} \tag{90}$$

$q$ measures the rate at which the expansion of the universe is slowing down. Current estimates get a negative value, meaning the universe expansion is not subduing but increasing. From (74) we can include second order effects into account and find that:

$$d_L = zT_0[1 - \frac{1}{2}(1 + q_0)z + \cdots]. \tag{91}$$

(88) is fine for nearby galaxies. But beyond 18th magnitude (90) has to be used. Notice that this latter one is a function of $q_0$.

Differentiating (64)

$$2\dot{R}\ddot{R} = -\frac{C}{R^2}\dot{R} + \frac{2}{3}\Lambda R\dot{R}, \tag{92}$$

and multiplying by $-R/2\dot{R}^3$ we get,

$$-\frac{R\ddot{R}}{\dot{R}^2} = \frac{C}{2R\dot{R}^2} - \frac{1}{3}\Lambda\frac{R^2}{\dot{R}^2}. \tag{93}$$

Then from (89), (63) and (87) we get:

$$q = \left(\frac{4}{3}\pi\rho - \frac{1}{3}\Lambda\right)/H^2 \tag{94}$$

Another important observable is $N$, the number of galaxies in a given volume. The volume is given by:

$$V = 4\pi R^3(t_0)\int_0^{r_1}\frac{r^2 dr}{(1-kr^2)^{1/2}}. \tag{95}$$

The number of galaxies in this volume is

$$N = Vn(t_0). \tag{96}$$

Is this number constant? We need a theory of galactic evolution. $H, q, \rho$ and $N$ play a crucial role in determining different models and possible evolutions for our universe.

We can go back to

$$\frac{dr}{(1-kr^2)^{1/2}} = -\frac{dt}{R(t)}$$

and we see that we can now put

$$\frac{dr}{(1-kr^2)^{1/2}} = -\frac{dt}{R(t)} = \frac{dz}{R_0 H(z)},$$

after using that $H = \frac{\dot{R}(t)}{R(t)}$ and also eq (72)

$$1 + z = \frac{R(t_0)}{R(t_1)}$$

And integrating assuming small $r$ and $z$ and working only to first order beyond the Euclidean relations:

$$d_L = R_0 r(1+z) = \left(\frac{z}{H_0}\right)\left[1 + \left(1 + \frac{1}{2}\frac{\dot{H}_0}{{H_0}^2}\right)z\right] + \cdots$$

If we can measure the luminosity distances and redshifts of a number of objects, then we can measure $\dot{H}_0$.

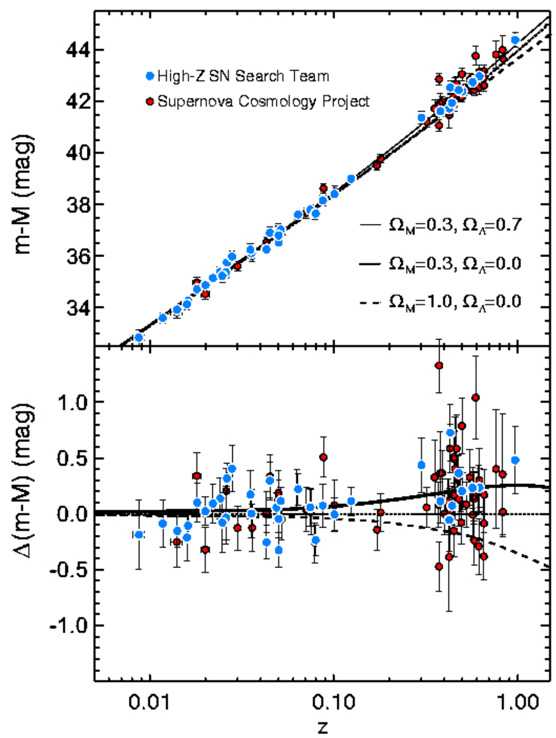
**The Universe is accelerating**

The way this is observed is by doing a plot of the luminosity distance against redshift for Type Ia supernovae.

These occur occur in binary systems in which one of the stars is a white dwarf while the other can vary from a giant star to an even smaller white dwarf.

This category of supernovae produces a consistent peak luminosity because of the uniform mass of white dwarfs that explode via the accretion mechanism (carbon-oxygen white dwarfs with a low rate of rotation are limited to below 1.38 solar masses).

The stability of this value allows these explosions to be used as standard candles to measure the distance to their host galaxies because the visual magnitude of the supernovae depends primarily on the distance.




$L_d$ vs $z$ for Type Ia supernovae

**Cosmological models**

*Flat case* $k = 0$

In this case we get:

$$\dot{R}^2 = C/R + \frac{1}{3}\Lambda R^2. \tag{97}$$

We assume $\Lambda > 0$ and introduce a new variable $u$:

$$u = \frac{2\Lambda}{3C} R^3. \tag{98}$$

Differentiating,

$$\dot{u} = \frac{2\Lambda}{C} R^2 \dot{R}, \tag{99}$$

and substituting in (97)

$$\dot{u}^2 = \frac{4\Lambda^2}{C^2} R^4 \left( \frac{C}{R} + \frac{1}{3}\Lambda R^2 \right) \tag{100}$$

$$= \frac{4\Lambda^2}{C} R^3 + \frac{4\Lambda^3}{3C^2} R^6 \tag{101}$$

$$= 6\Lambda u + 3\Lambda u^2 \tag{102}$$

$$= 3\Lambda(2u + u^2). \tag{103}$$

or,

$$\dot{u} = (3\Lambda)^{\frac{1}{2}} (2u + u^2)^{\frac{1}{2}}. \tag{104}$$

This equation can be integrated by parts...
Assuming $R = 0$ when $t = 0$, then $u = 0$ and we have,

$$\int_0^u \frac{du}{(2u + u^2)^{\frac{1}{2}}} = \int_0^t (3\Lambda)^{1/2} dt = (3\Lambda)^{1/2} t. \tag{105}$$

Completing squares and making $v = u + 1$ and $\cosh w = v$

$$\int_0^u \frac{du}{[(u+1)^2 - 1]^{\frac{1}{2}}} = \int_1^v \frac{\sinh w dw}{(\cosh^2 w - 1)^{1/2}} \tag{106}$$

$$= \int_0^w dw = w. \tag{107}$$

And going back to $R$,

$$R^3 = \frac{3C}{2\Lambda} \left[ \cosh(3\Lambda)^{1/2} t - 1 \right]. \tag{108}$$

If $\Lambda < 0$ we can introduce:

$$u = -\frac{2\Lambda}{3C} R^3 \tag{109}$$

and then we can get,

$$R^3 = \frac{3C}{2(-\Lambda)} \left\{ 1 - \cosh\left[3\left(-\Lambda\right)\right]^{\frac{1}{2}} t \right\}. \tag{110}$$

If $\Lambda = 0$

$$\dot{R} = \left(\frac{C}{R}\right)^{1/2}. \tag{111}$$

Direct integration gives,

$$R = \left(\frac{9}{4} C t^2\right)^{\frac{1}{3}}. \tag{112}$$

This is the **Einstein-de Sitter** model. The Hubble parameter is:

$$H(t) = \dot{R}/R = 2/(3t). \tag{113}$$

The deceleration parameter is:

$$q(t) = -R\ddot{R}/\dot{R}^2 = \frac{1}{2}. \tag{114}$$

At the beginning of the expanding universe, $R$ is small and $C/R$ dominates. So for small $t$

$$\dot{R}^2 \sim C/R, \tag{115}$$

integrating,

$$R \sim \left(\frac{9}{4} C t^2\right)^{\frac{1}{3}}. \tag{116}$$

In early stages all models regardless of the value of $\Lambda$ expand like $t^{2/3}$

**Models with vanishing cosmological constant**

$$\dot{R}^2 = C/R - k, \tag{117}$$

We consider two cases $k = +1$ and $k = -1$ **k=+1** (116) becomes

$$\dot{R}^2 = C/R - 1, \tag{118}$$

We define:

$$u^2 = R/C, \tag{119}$$

And then $2u\dot{u} = \dot{R}/C$, and substituting in (116):

$$\dot{u}^2 = \frac{\dot{R}^2}{4C^2u^2} = \frac{1}{4C^2u^2}\left(\frac{C}{R} - 1\right) = \frac{1}{4C^2u^2}\left(\frac{1}{u^2} - 1\right) \tag{120}$$

The equation is separable if we take positive square roots,

$$2\int_0^u \frac{u^2}{(1-u^2)^{\frac{1}{2}}} du = \frac{1}{C}\int_0^t dt = \frac{t}{C} \tag{121}$$

We can evaluate the $u-$integral, we make $u = sin\theta$.

$$2\int_0^u \frac{u^2}{(1-u^2)^{\frac{1}{2}}} du = 2\int_0^\theta \frac{sin^2\theta cos\theta d\theta}{(1-sin^2\theta)^{1/2}} = \tag{122}$$

$$sin^{-1}u - u(1-u^2)^{1/2} \tag{123}$$

which back to $R$ yields,

$$C[\sin^{-1}(R/C)^{1/2} - (R/C)^{1/2}(1-R/C)^{1/2}] = t. \tag{124}$$

In the case **k=-1** we get...

$$C[(R/C)^{1/2}(1+R/C)^{1/2} - \sinh^{-1}(R/C)^{1/2}] = t. \tag{125}$$

The case $\lambda = 0$, $k = 0$ is the Einstein-de Sitter model of eq (115) The Hubble and deceleration parameters are,

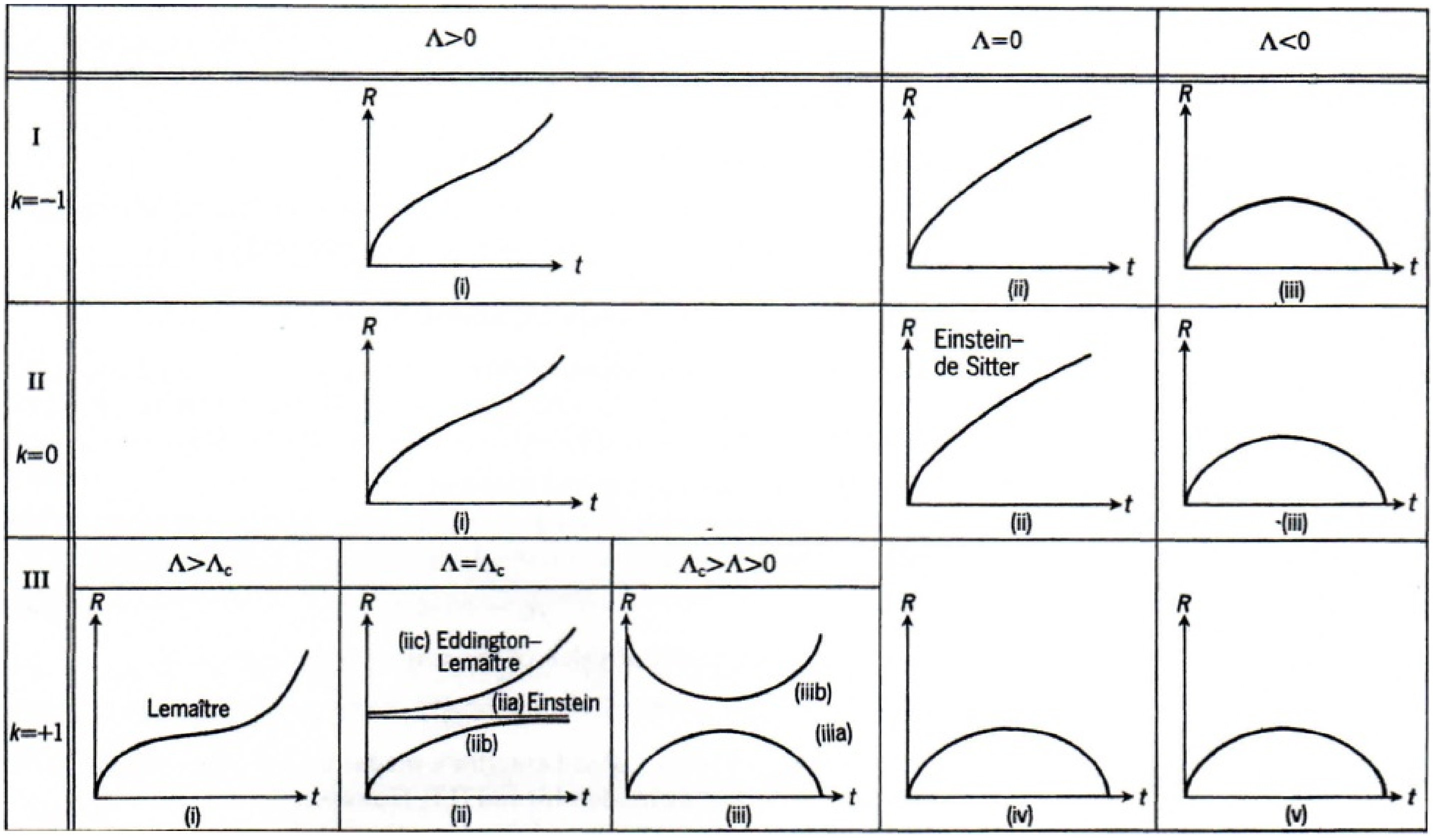
$$H = C^{-1}(R/C)^{-3/2}(1-R/C)^{1/2} \tag{126}$$

$$q = \frac{1}{2}(1-R/C)^{-1} \tag{127}$$

where $R$ is a function of (t) implicitly from (123).
We can write (117) as $\dot{R}^2 = G(R)$, and then we get

$$G(R) = C/R - k, \tag{128}$$

We can see that $k = +1$ has a local minimum, while the other models grow without bound.
If $k = -1$, and $t \to \infty$ we'll get $\dot{R}^2 \sim 1$, and then $R \sim t$.




Friedmann models

**Friedman-Robertson Walker universes**

Using (54) and assuming that the matter content is a perfect fluid, we can look at $T^{\mu\nu}{}_{;\nu} = 0$. The only non trivial component is $\mu = 0$ and we get:

$$\frac{d}{dt}\left(\rho R^3\right) = -p\frac{d}{dt}\left(R^3\right) \tag{129}$$

where $R(t)$ is the cosmological expansion factor. $R^3$ is proportional to the volume of the fluid and then the left hand of (128) is the rate of change of energy, and the right hand is the work it does as it expands $(-pdV)$.
In a matter dominated universe we have $p = 0$ and then

$$\frac{d}{dt}\left(\rho R^3\right) = 0 \tag{130}$$

In a radiation dominated era, $p = \frac{1}{3}\rho$

$$\frac{d}{dt}\left(\rho R^3\right) = -\frac{1}{3}\rho\frac{d}{dt}\left(R^3\right) \tag{131}$$

or

$$\frac{d}{dt}\left(\rho R^3\right) = 0 \tag{132}$$

In Einstein's els the only two components non zero are $G_{tt}$ and $G_{rr}$, so only one component survives (due to Bianchi's identities):

$$G_{tt} = 3(\frac{\dot{R}}{R})^2 + 3k/R^2 \tag{133}$$

So besides (129) or (131) we have Einstein's eq with a cosmological constant $\Lambda$

$$G_{tt} + \Lambda g_{tt} = 8\pi T_{tt} \tag{134}$$

We can think then of $\Lambda$ as the energy density and pressure of a fluid

$$\rho_\Lambda = \Lambda/8\pi, \quad p_\Lambda = -\rho_\Lambda \tag{135}$$

$\rho_\Lambda$ is called the dark energy.
Then Einstein's eqs can be written,

$$\frac{1}{2}\dot{R}^2 = -\frac{1}{2}k + \frac{4}{3}\pi R^2(\rho_m + \rho_\Lambda) \tag{136}$$

From (128) and the time derivative of (135) we get

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}(\rho + 3p), \tag{137}$$

where $\rho$ and $p$ are the total energy and density pressure for both matter and dark energy.

**Thermodynamics in Cosmology**

The evolution of matter, radiation and even vacuum is contained in the behavior of $R$ as a function of time $t$. This is one of the key features of an homogeneous model of the universe.

For any change $d(\Delta V)$ in a volume $\Delta V$ a change in the energy of the universe is given by:

$$d(\Delta E) = -pd(\Delta V) \tag{138}$$

where

$$\Delta E = \rho \Delta V \tag{139}$$

and $\rho$ is the energy density. $\Delta x, \Delta y, \Delta z$ define a volume where the number of particles (galaxies) remain fixed due to the fact that we are utilizing comoving coordinates. But $\Delta V_{coord} = \Delta x \times \Delta y \times \Delta z$ is not the physical size of the volume. This can be calculated explicitly using

$$dV = \sqrt{g} dx^1 dx^2 dx^3 \tag{140}$$

and

$$ds^2 = R^2(t)(dx^2 + dy^2 + dz^2) \tag{141}$$

from where we obtain

$$\Delta V = R^3(t) \Delta V_{coord} \tag{142}$$

Using this last formula in (138) and dividing by $dt$ we get

$$\frac{d}{dt}(\rho R^3 \Delta V_{coord}) = -p \frac{d}{dt}(R^3 \Delta V_{coord}) \tag{143}$$

But $)\Delta V_{coord}$ is independent of time (comoving system of coordinates), and then

$$\frac{d}{dt}(\rho(t) R^3(t)) = -p(t) \frac{d}{dt}(R^3(t)) \tag{144}$$

This is the First Law of Thermodynamics for a homogeneous isotropic cosmology. We can investigate how it applies for the three kinds of energy we consider in a FRW model. We will assume that it applies to each form separately.

**Matter**

We can assume that matter in galaxies can be assumed as a pressure-less gas. Then (144) becomes

$$\frac{d}{dt}(\rho_m(t) R^3(t)) = 0 \tag{145}$$

which shows the conservation of mass and as a consequence a time evolution:

$$\frac{d}{dt}\rho_m(t) = \rho_m(t_0)\left(\frac{R(t_0)}{R(t)}\right)^3 \tag{146}$$

The time evolution of the matter density is entirely determined by the scale factor of the universe.

**Radiation**

For a gas of blackbody radiation at temperature $T$ its energy and pressure are:

$$p_r = \frac{1}{3}\rho_r \tag{147}$$

and the energy density

$$\rho_r = g\frac{\pi^2}{30}\frac{(k_B T)^4}{(\hbar c)^3} \tag{148}$$

where $k_B$ is the Boltzmann's constant ($k_B = 1.38 \times 10^{-16}$ erg/K) and $g$ is the number of degrees of freedom of the massless particles making up the radiation. $g = 2$ for photons (2 polarizations). For the case of neutrinos $g \approx 3.4$ when including the three species and for $k_B T <\sim 1MeV$ which we can use here.

Using (147) in (144 ) we get after integration

$$\rho_r(t) = \rho_r(t_0)\left(\frac{R(t_0)}{R(t)}\right)^4 \tag{149}$$

or following (148)

$$T(t) = T(t_0)\left(\frac{R(t_0)}{R(t)}\right) \tag{150}$$

We can see that the time dependence of the radiation energy density and its temperature are determined by the scale factor. Temperature is inversely proportional to the scale factor.
Many large scale properties of the matter in the universe, for example the primordial abundance of the elements, can be understood as determined by the cool down from an initial thermal equilibrium at very high temperature.

Matter dominates radiation now, but the opposite was true in the early universe. For any density value at the current time there was an earlier value when $R(t)$ was smaller and $\rho_r$ bigger than $\rho_m$. At that moment the universe was radiation dominated. There are roughly $10^{11}$ galaxies in the portion of the universe that is accesible to observations today. This would correspond to a density of approximately

$$\rho_{visible}(t_0) \sim 10^{-31} g/cm^3. \tag{151}$$

The density of the cosmic background radiation with a temperature of $2.725K$ is

$$\rho_r(t_0) \sim 10^{-34} g/cm^3. \tag{152}$$

From (146) and (149) then the universe was radiation dominated when

$$R(t_0)/R(t) \sim 10^3 \tag{153}$$

i.e. when the universe was 1/1000 its current size.

**Vacuum**

Let's concentrate on a vacuum energy that is (i) constant in space and time, and (ii) positive as indicated by present observations. The first law of thermodynamics implies that

$$p_v = -\rho_v \tag{154}$$

(a negative pressure is like tension in a rubber band: it requires work to expand the volume rather than work to compress it). For historical reasons:

$$\rho_v = \frac{c^4\Lambda}{8\pi G} \tag{155}$$

where $\Lambda$ is the cosmological constant and has dimensions of a inverse square length. The long term evolution our universe seems to be dominated by the vacuum energy.

**Evolution of flat FRW models**

The Friedman equation for a flat universe ($k = 0$) is:

$$\dot{R}^2 - \frac{8\pi\rho}{3}R^2 = 0 \tag{156}$$

This can be thought as a balance of the potential energy of gravitational self-attraction by the kinetic energy of the expansion of a flat FRW universe. If we divide it by $R^2(t_0)$ we can obtain an equation relating the Hubble constant today to the current density of the universe:

$$H_0^2 - \frac{8\pi\rho_0}{3} = 0 \tag{157}$$

The present density of a flat FRW model is called the critical density and has the value

$$\rho_{crit} \equiv \frac{3H_0^2}{8\pi} = 1.88 \times 10^{-29} h^2 g/cm^2 \tag{158}$$

where $h \equiv H_0/[(km/s)/Mpc] \approx .7 \pm .1$.

This total density can be thought of made up of the densities of all matter, radiation and vacuum energies. The relative fractions are customarily written:

$$\Omega_m \equiv \frac{\rho_m(t_0)}{\rho_{crit}}, \qquad \Omega_r \equiv \frac{\rho_r(t_0)}{\rho_{crit}}, \qquad \Omega_\Lambda \equiv \frac{\rho_\Lambda(t_0)}{\rho_{crit}}, \tag{159}$$

where $\Omega_m + \Omega_r + \Omega_\Lambda = 1$ for these flat models.

It is also clear that (156) determines $R(t)$ only up to a multiplicative constant. For what it follows we will normalize $R(t_0) = 1$. With this normalization we get

$$\rho(R) = \rho_{crit}\left(\Omega_\Lambda + \frac{\Omega_m}{R^3} + \frac{\Omega_r}{R^4}\right), \tag{160}$$

Notice that although this can be calculated when $R(t_0) = 1$, it is valid for all times due to the functional behavior of the different densities as powers of $R$. Then we can write (156) as

$$\frac{1}{2H_0^2}\dot{R}^2 + U_{eff}(R) = 0 \tag{161}$$

where $U_{eff}(R)$ is given by

$$U_{eff}(R) = \frac{1}{2}\left(\Omega_\Lambda R^2 + \frac{\Omega_m}{R} + \frac{\Omega_r}{R^2}\right), \tag{162}$$

This is in many regards an energy conservation equation.
Solving it to study the evolution of the flat FRW model for each energy content gives:

• Matter dominated: $\Omega_m = 1, \Omega_r = 0, \Omega_\Lambda = 0$:

$$R(t) = \left(\frac{t}{t_0}\right)^{2/3}, \tag{163}$$

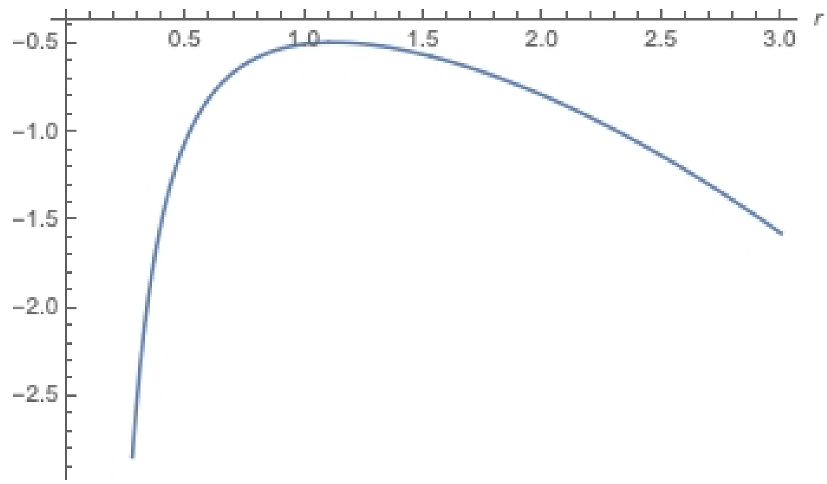
• Radiation dominated: $\Omega_m = 0, \Omega_r = 1, \Omega_\Lambda = 0$:

$$R(t) = \left(\frac{t}{t_0}\right)^{1/2}, \tag{164}$$

```
In[12]:= Plot[Ueff, {r, 0, 3}, Axes → True, AxesLabel → {r, Ueff}]
```

$$\frac{1}{6}\left(-r^2 - \frac{1}{r} - \frac{1}{r^2}\right)$$

Out[12]=



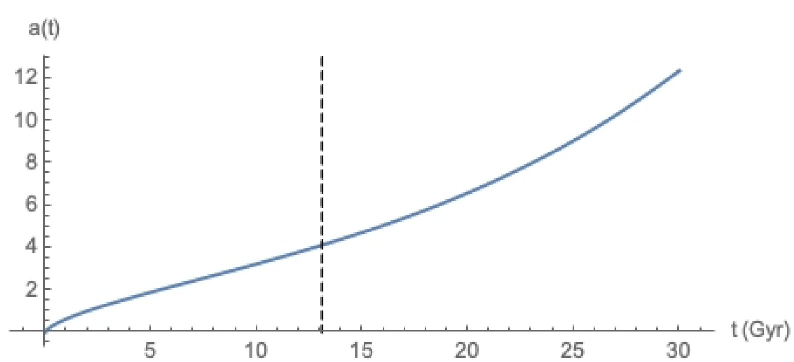

The effective potential $U_{eff}$ for a FRW model with equal amounts $\Omega_m = \Omega_r = \Omega_\Lambda = 1/3$. The universe starts with a big bang at $r = 0$, decelerates until $r \approx 1$ (today) and then accelerates for ever, when it becomes vacuum dominated.

• Vacuum dominated: $\Omega_m = 0, \Omega_r = 0, \Omega_\Lambda = 1$:

$$R(t) = e^{H(t-t_0)}, \tag{165}$$

where $H^2 \equiv \frac{8\pi\rho_\Lambda}{3} = \frac{\Lambda}{3}$. In all three cases the universe expands as time increases. In the radiation and matter dominated cases, the universe begins with a singularity ($R = 0$ at $t = 0$). Notice that this is a physical singularity: the density energy a physical quantity becomes infinity at $t = 0$. In the vacuum dominated case $R \to 0$ at $t = -\infty$. The nature of this behavior is less clear: the universe has some matter and radiation in it and it had a big-bang singularity.

Out[84]=



The different stages of evolution of a FRW flat model with equal amounts $\Omega_m = \Omega_r = \Omega_\Lambda = 1/3$. The potential is shown in the previous figure. It is first radiation dominated, then matter dominated and finally vacuum dominated. The vertical line shows the present time with $R(t_0) = 1$. Both plots have use the Mathematica notebooks provided by Leonard Parker's book.

**The first 3 minutes**

Big Bang nucleosynthesis (also known as primordial nucleosynthesis, BBN) explains the production of light nuclei, deuterium, $^3He$, $^4He$, $^7Li$, between 0.01s and 200s in the lifetime of the universe.

Elements heavier than lithium are thought to have been created later in the life of the Universe by stellar nucleosynthesis, through the formation, evolution and death of stars.

BBN assumes a homogeneous plasma, at a temperature corresponding to 1 MeV, consisting of electrons annihilating with positrons to produce photons. In turn, the photons pair to produce electrons and positrons: $e^+e^- \leftrightarrow \gamma\gamma$. These particles are in equilibrium. A similar number of neutrinos, also at 1 MeV, have just dropped out of equilibrium at this density. Finally, there is a very low density of baryons (neutrons and protons). The BBN model follows the nuclear reactions of these baryons as the temperature and pressure drops due to expansion of the universe.

The basic model makes two simplifying assumptions:

1. until the temperature drops below 0.1 MeV only neutrons and protons are stable and
2. only isotopes of hydrogen and of helium will be produced at the end.

These assumptions are based on the intense flux of high energy photons in the plasma. Above 0.1 MeV every nucleus created is blasted apart by a photon. Thus the model first determines the ratio of neutrons to protons and uses this as an input to calculate the hydrogen, deuterium, tritium, and $^3He$.

Around $kT \approx 1$ MeV, the density of neutrinos drops, and reactions like $n + e^+ \leftrightarrow p + \overline{\nu}_e$ which maintained neutron and proton equilibrium, slow down. The neutron-to-proton ratio decreases to around 1/7. As the temperature and density continue to fall, reactions involving combinations of protons and neutrons shift towards heavier nuclei. These include $p + n \to \mathrm{D} + \gamma,\ \mathrm{D} + \mathrm{D} \to n + {}^3\mathrm{He},\ {}^3\mathrm{He} + \mathrm{D} \to p + {}^4\mathrm{He}$. Due to the higher binding energy of He, the free neutrons and the deuterium nuclei are largely consumed, leaving mostly protons and helium.

The fusion of nuclei occurred between roughly 10 seconds to 20 minutes after the Big Bang; this corresponds to the temperature range when the universe was cool enough for deuterium to survive, but hot and dense enough for fusion reactions to occur at a significant rate.

The key parameter which allows one to calculate the effects of Big Bang nucleosynthesis is the baryon/photon number ratio, which is a small number of order $6 \times 10^{-10}$. This parameter corresponds to the baryon density and controls the rate at which nucleons collide and react; from this it is possible to calculate element abundances after nucleosynthesis ends. Although the baryon per photon ratio is important in determining element abundances, the precise value makes little difference to the overall picture. Without major changes to the Big Bang theory itself, BBN will result in mass abundances of about 75% of hydrogen-1, about 25%

helium-4, about $0.01\%$ of deuterium and helium-3, trace amounts (on the order of $10^{-10}$) of lithium, and negligible heavier elements. That the observed abundances in the universe are generally consistent with these abundance numbers is considered strong evidence for the Big Bang theory.

The primordial abundance of the elements is established when the temperature after the Big-Bang drops below $\sim 0.1$ MeV and the thermonuclear reactions which can alter the relative abundance of the existing elements stops.

Let's calculate when this primordial abundance is set. The early universe is radiation dominated. The energy density is well approximated as a function of temperature with (148)

$$\rho_r = g\frac{\pi^2}{30}\frac{(k_B T)^4}{(\hbar c)^3}$$

where a good phenomenological take for $g$ is $g = 3.4$. The scale facto is $R \sim t^{1/2}$. With the equation above and (156) we get

$$\frac{1}{4t^2} = \frac{8\pi}{3}\rho(t) = 2.75g\frac{1}{G\hbar/c^3}\left(\frac{T}{(G\hbar/c^3)^{1/2}}\right)^4. \tag{166}$$

In units of time in seconds and temperature in MeV, we get

$$t = 1.3\left(\frac{1MeV}{T}\right)^2 s \tag{167}$$

which gives when $T = 0.1$ MeV a $t = 130$ which rounding up is approximately the first 3 minutes.

**Age and the Hubble constant in the flat FRW model**

If we assume we live in a flat FRW universe with a metric

$$ds^2 = -dt^2 + R^2(t)(dx^2 + dy^2 + dz^2) \tag{168}$$

If we assume that matter is the dominant form of energy $R(t) \propto (t)^{2/3}$ and the equation connecting with the Hubble time is

$$H_0 \equiv H(t_0) \equiv \frac{\dot{R}(t_0)}{R(t_0)}. \tag{169}$$

which gives

$$t_0 = \frac{2}{3H_0} = \frac{2}{3}t_H. \tag{170}$$

Assuming a value of $H_0 = 72$ (km/s)/Mpc the age opf the Universe is approximately 9 Gyr. But the age of the oldest stars in our galaxy is approximately 12 Gyr. This indicates that the flat FRW is not a perfect fit for modeling the evolution of our universe.

**General Solution of the Friedman Equation**

It is appropriate to use a dimensionless scale factor:

$$\bar{R} \equiv R(t)/R_0 \tag{171}$$

This is directly related to the redshift $z$ of radiation coming from comoving galaxies at the time $t$ by $\bar{R} = 1/(1+z)$. Similarly the Hubble time can be used to define a dimensionless measure of time:

$$\bar{t} \equiv t/t_H = H_0 t \tag{172}$$

The critical density defined as (158)

$$\rho_{crit} \equiv \frac{3H_0^2}{8\pi} = 1.88 \times 10^{-29} h^2 g/cm^2$$

can be used to scale densities. For example

$$\rho_r = \rho_{crit}\Omega_r/(\bar{R}(t))^4, \tag{173}$$

We can even introduce an $\Omega_c$ to quantify the curvature defining

$$\Omega_c = -k/(H_0 R_0)^4, \tag{174}$$

With this definition the Friedman equation reads at the present time

$$\Omega_\Lambda + \Omega_m + \Omega_r + \Omega_c = 1, \tag{175}$$

Of course this could not work if we are dealing with a closed universe model where $\Omega_c < 0$. The rescaled Friedman equation reads now:

$$\frac{1}{2}\left(\frac{d\bar{R}}{dt}\right)^2 + U_{eff}(\bar{R}) = \frac{\Omega_c}{2}. \tag{176}$$

where the effective potential energy is

$$U_{eff}(\bar{R}) \equiv -\frac{1}{2}\left(\Omega_\Lambda \bar{R}^2 + \frac{\Omega_m}{\bar{R}} + \frac{\Omega_r}{\bar{R}^2}\right), \tag{177}$$

and $\Omega_c$ is given in terms of the other $\Omega$'s by (175). Equations (176) and (177) reduce to (161) and (162) for a flat FRW model.

To construct a general FRW cosmological model from here the procedure is:
(1) Specify the four parameters $H_0, \Omega_\Lambda, \Omega_m, \Omega_r$. (2) Use the last three to solve (176) for $\bar{R}(\bar{t})$ by writing

$$d\bar{R}\left(\Omega_c - 2U_{eff}(\bar{R})\right)^{-1/2} = d\bar{t} \tag{178}$$

Then we have to undo the rescaling of $H_0$ to go back to $t$ from $\bar{t}$ and find the value of $R_0$ from (174). The result for $R(t)$ is

$$R(t) = \frac{1}{H_0|\Omega_c|^{1/2}}\bar{R}(H_0 t). \tag{179}$$

A FRW cosmology is consequently determined by the four cosmological parameters:

$$\boxed{H_0, \qquad \Omega_r, \qquad \Omega_m, \qquad \Omega_\Lambda,} \tag{180}$$

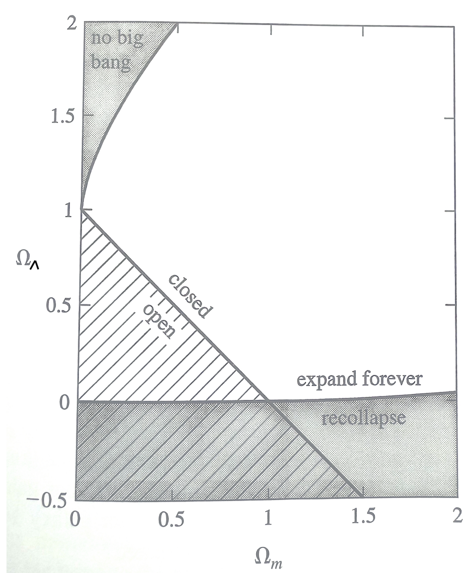
These specify the past, present and future of the universe. To determine them from observation is the goal of theoretical cosmology.

**The age of the universe as a function of the cosmological parameters**

From (172) the age of the universe is

$$t_0 = \frac{1}{H_0}\bar{t}_0\left(\Omega_r, \Omega_m, \Omega_\Lambda\right) \tag{181}$$

The function $\bar{t}_0\left(\Omega_r, \Omega_m, \Omega_\Lambda\right)$ is dimensionless and is the value of $\bar{t}$ where $\bar{R}(\bar{t}_0) = 1$.
This function can be obtained integrating (176).




The different stages of evolution of FRW models in the $\Omega_m - \Omega_\Lambda$ plane.