Lesson 12
Cosmology
or The Natural History of Everything

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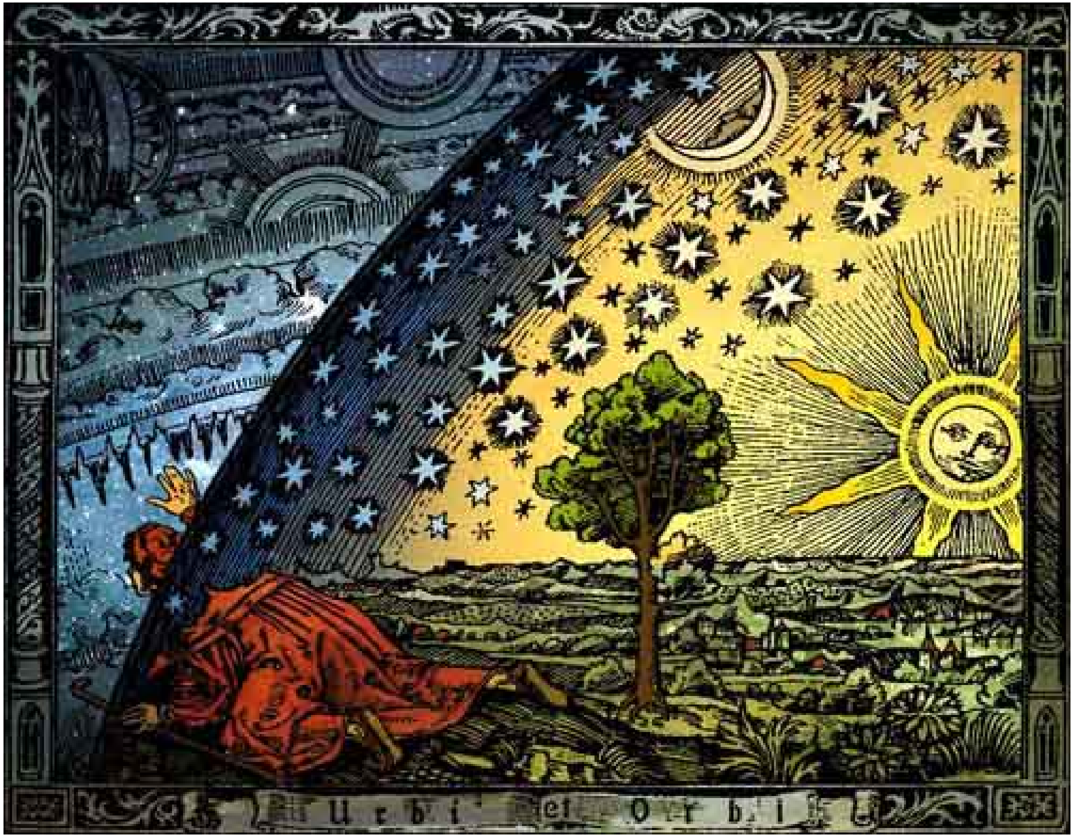
What is Cosmology?

most of the material follows d’Inverno’s ”Introducing Einstein’s Relativity”, although I kept the signature we have used throughout the course and not the one the author utilizes in his book.

I follow Schutz in discussing luminosity distance and the expansion of the universe. Similarly when I introduce dark energy. The layout for this part of the course will follow the discussion of the following issues or questions.

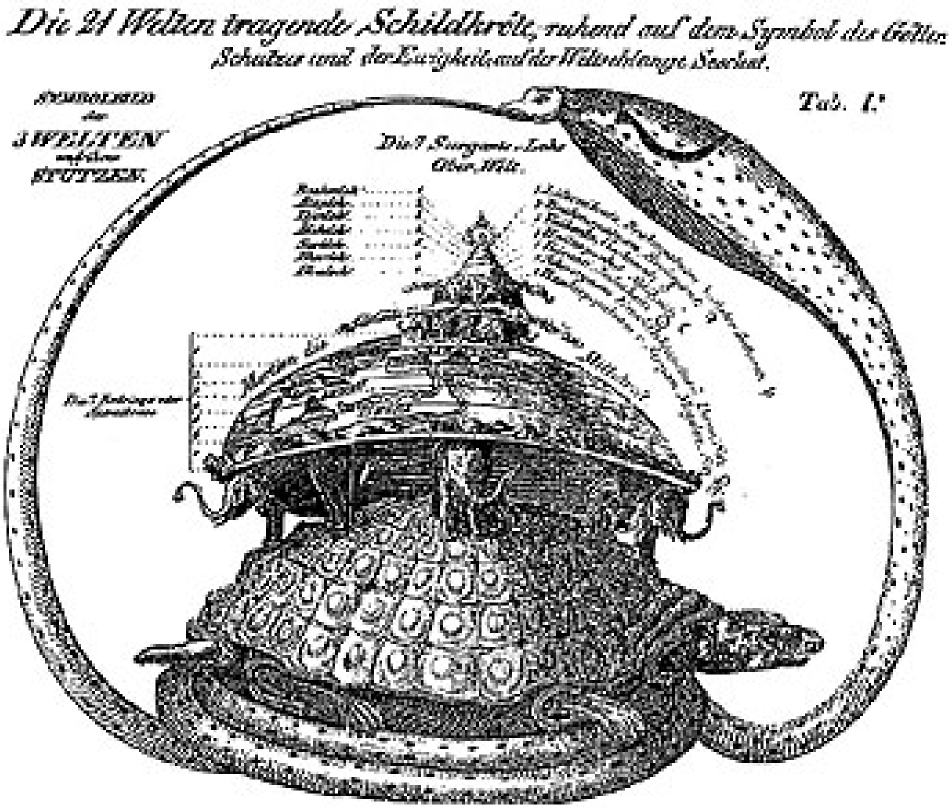
- How does everything fit together?
- Olbers paradox
- Newtonian Cosmology
- The Cosmological Principle
- Weyl's postulate
- Relativistic Cosmology



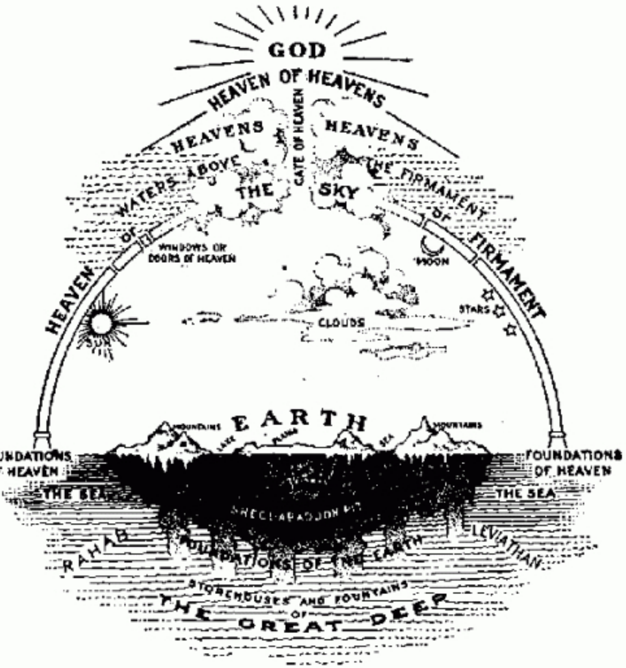
The ancient world of the Westerners

The Cosmos according to the Incas

An according to the Mayans



The world for the Ancient Hindus



THE ANCIENT HEBREW CONCEPTION OF THE UNIVERSE

OF THE UNIVERSE

TO ILLUSTRATE THE ACCOUNT OF CREATION AND THE FLOOD

The world for the Ancient Hebrew

Olbers' paradox

In 1826 Olbers postulated his famous paradox:

- How come the sky is so dark if it's filled with stars in an infinite universe?

It is not difficult to see that if we look at the center of a shell of radius r with a total luminosity l provided by the average luminosity of the stars contained multiplied by the number of stars inside this volume the intensity of the light produced at the center of the shell will be this total luminosity divided by the area of it, i.e.:

$$\frac{(4\pi r^2 dr)l}{4\pi r^2} = ldr \tag{1}$$

We clearly get the total intensity at P by integrating over all the shells around P up to infinity:

$$\int_0^\infty ldr = \infty! \tag{2}$$

But the sky is dark at night! The paradox is that a static, infinitely old universe with an infinite number of stars distributed in an infinitely large space would be bright rather than dark. We could have taken into consideration: Absorption of light by stars in the line of sight. Olbers postulated the existence of a tenuous gas which would absorb the radiation (this is an inconsistent argument from the point of view of Thermodynamics). The expansion of the Universe would definitely be able to provide an explanation for the paradox. For a more complete explanation of the paradox and interesting alternative resolutions see:

$http : //en.wikipedia.org/wiki/Olbers'\_paradox$

(excellent account, including historical precedents).

The Homogeneous Isotropic Cosmological Model

At any given instant of time:

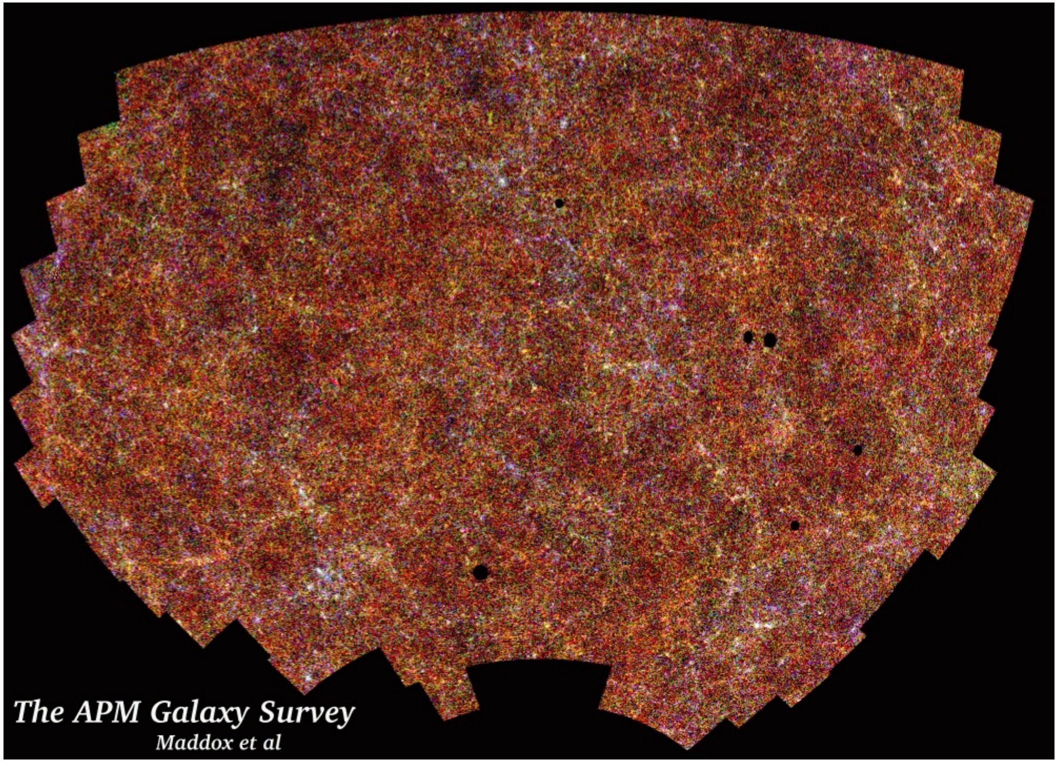
- **Homogeneity:** At different space points all observable quantities are the same. This means no matter which region we look at they all look similar at a given scale. We find the same proportion of spiral galaxies, elliptical galaxies, irregular galaxies. The clustering of galaxies at larger scales is observed to be distributed equally along the universe. No preferred point or location in space.

- **Isotropy:** At any space point all directions are equivalent. No special vantage point. No rotation (that would indicate an axis and then a preferred direction).

We could investigate:

- Relativistic models (i.e. using the general theory of relativity).

- Newtonian gravity.



Galaxy Survey 30° across with a million galaxies up to a distance of 2 billion light years.

Newtonian Cosmology

- A spherically symmetric distribution of matter does not give rise to a gravitational force inside a spherical cavity concentric to the matter distribution.

- We can assume a distribution of velocities for the particles that make up this fluid: $v = Hr$

Edwin Hubble discovered his law when studying distance to galaxies and their spectra redshift in 1929.

Features of our model:

- The matter at the origin of the coordinate system is at rest.
- Only the assumed type of velocity distribution is isotropic and homogeneous.
- An observer moving along with the particles sees all neighboring particles receding.

Evolution of the model:

$$v = Hr \qquad\qquad \frac{dr}{dt} = Hr \tag{3}$$

$$r_{AB}(t) = r_{AB}(t_0) \int_{t_0}^{t} H(t)dt \tag{4}$$

Density evolution

Let's assume a mass M,volume of radius R, then:

$$\rho = \frac{M}{4/3\pi R^3} \qquad \rightarrow \qquad \frac{d\rho}{dt} = \frac{-3M}{4/3\pi R^4}\frac{dR}{dt} \tag{5}$$

And if we substitute:

$$dR/dt = \nu = HR \tag{6}$$

$$\frac{d\rho}{dt} = \frac{-3M}{4/3\pi R^4}HR = -3\rho H \tag{7}$$

And we have:

$$\frac{d\rho}{dt} = -3\rho H \tag{8}$$

How does the velocity change?

We now will use Newton's gravity:

$$\frac{d\nu}{dt} = a = -\frac{GM}{r^2} = -\frac{G\frac{4}{3}\pi\rho r^3}{r^2} = -\frac{4}{3}\pi G\rho r \quad (9)$$

How do H and ρ evolve?

Using (6), $\rho = \frac{M}{4/3\pi R^3}$ and $a = \frac{d^2R}{dt^2}$ we get:

$$\frac{d^2R}{dt^2} = \frac{d}{dt}(HR) = \frac{dH}{dt}R + H\frac{dR}{dt} = R\frac{dH}{dt} + HHR \quad (10)$$

From where we have:

$$\frac{dH}{dt} = -H^2 - \frac{4}{3}\pi G\rho \tag{11}$$

$$\frac{d\rho}{dt} = -3\rho H \tag{12}$$

Eqs (11) and (12) form a complete system of equations.

We can now multiply (9) by $\frac{dR}{dt}$

$$\frac{d^2R}{dt^2}\frac{dR}{dt} = -\frac{GM}{r^2}\frac{dR}{dt} \tag{13}$$

which yields:

$$\frac{1}{2}\frac{d}{dt}(\dot{R}^2) = GM\frac{d}{dt}\left(\frac{1}{R}\right) \tag{14}$$

$$\frac{d}{dt}\left[\frac{1}{2}(\dot{R}^2) - \frac{GM}{R}\right] = 0 \tag{15}$$

$$\frac{1}{2}(\dot{R}^2) - \frac{GM}{R} = constant \tag{16}$$

We can calculate the constant at $t = t_0$

$$A = \frac{1}{2}(H_0^2 R_0^2) - \frac{4}{3}\pi G \rho_0 R_0^2 \tag{17}$$

and using

$$M = \frac{4}{3}\pi \rho_0 R_0^3 \tag{18}$$

$$\left(\frac{dR}{dt}\right)^2 = \frac{8}{3}\pi G \rho_0 \frac{R_0^3}{R} - \frac{8}{3}\pi G R_0^2 \left(\rho_0 - \frac{3H_0^2}{8\pi G}\right) \tag{19}$$

where we can define a critical density:

$$\rho_c = \frac{3H_0^2}{8\pi G} \tag{20}$$

$$\left(\frac{dR}{dt}\right)^2 = \frac{8}{3}\pi G\rho_0\frac{R_0^3}{R} - \frac{8}{3}\pi G R_0^2\left(\rho_0 - \rho_c\right) \tag{21}$$

Now we can do a qualitative analysis:

- $dR/dt > 0$
- R increases with time

Then in the past $\frac{8}{3}\pi G\rho_0\frac{R_0^3}{R}$ was larger and also $\frac{dR}{dt}$ was large. So in the past should have been a time when $\rightarrow$

- $R = 0$

- $\frac{dR}{dt} = +\infty$

This is the **Big Bang!**

But the future depends on $(\rho - \rho_c)$.

We can write (21) defining two arbitrary constants:

$$\left(\frac{dR}{dt}\right)^2 = \frac{B}{R} - C\left(\rho_0 - \rho_c\right) \tag{22}$$

As R grows: If $\rho_0 > \rho_c$
and R is very small but $1/R$ grows until

$$R = \frac{B}{C(\rho_0 - \rho_0)} \tag{23}$$

And then $dR/dt = 0$ and the expansion stops!. But if $\rho_0 < \rho_c$
$dR/dt > 0$ and the expansion continues forever!

$$\frac{dR}{dt} = [C(\rho_c - \rho_0)]^{\frac{1}{2}} \tag{24}$$

And then if $\rho_0 = \rho_c$

$$\left(\frac{dR}{dt}\right)^2 = \frac{B}{R} \quad \rightarrow \quad R(t) = Dt^{2/3} \tag{25}$$

Relativistic Cosmology

Three postulates are the basis of RC:

- the cosmological principle: on large scale the universe looks the same to any observer (the universal Copernican Principle).

- Weyl's postulate: the universe can be represented by a perfect fluid, where the particles of the fluid are the galaxies.

- general relativity

Weyl's postulate can be expressed mathematically saying that there is a time, i.e. the proper time comoving with the galaxies. i.e. the galaxies move on time-like geodesics defining orthogonal hypersurfaces of constant coordinates. This orthogonality can be expressed:

$$ds^2 = dt^2 - h_{ij}dx^i dx^j \tag{26}$$

t is the cosmic time. The world map is the series of events on the surfaces of simultaneity (same t). The world picture is the set of events an observer sees in her past light cone at a given cosmic time. Due to the fact that we require isotropy and homogeneity we need to require that the spatial part of the metric be conformal in time, i.e. that the metric is multiply by an overall factor depending of time:

$$h_{ij} = S^2(t)g_{ij}(x^k)dx^i dx^j \tag{27}$$

The ratio of two values of S at different times is the magnification factor (scale factor). We will also require that the curvature at each point be constant, i.e. given a time slice the curvature of the surface has to be constant otherwise the isotropy and homogeneity will be lost. It can be shown that spaces of constant curvature are defined:

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{28}$$

where K is the constant curvature. Since the 3-space is isotropic about every point, it must be spherically symmetric. We can use the 3-space metric defined from (21) and (22) in Lesson 11:

$$d\sigma^2 = g_{ij}dx^i dx^j = e^\lambda dr^2 + r^2 d\Omega^2, \qquad (29)$$

where $\lambda = \lambda(r)$. The non-vanishing components of the RIcci tensor are:

$$R_{11} = \lambda'/r, R_{22} = cosec^2\theta R_{33}, \qquad (30)$$

$$R_{33} = 1 + \frac{1}{2} r e^{-\lambda}\lambda' - e^{-\lambda}. \qquad (31)$$

Condition (28) yields:

$$\lambda'/r = 2Ke^\lambda, 1 + \frac{1}{2} r e^{-\lambda}\lambda' - e^{-\lambda} = 2Kr^2. \qquad (32)$$

The solutions is:

$$e^{-\lambda} = 1 - Kr^2. \tag{33}$$

This gives us the metric for the 3-space of constant curvature:

$$d\sigma^2 = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2, \tag{34}$$

It is more convenient to define:

$$r = \frac{\bar{r}}{(1 + \frac{1}{4}K\bar{r}^2)}, \tag{35}$$

and the metric becomes:

$$d\sigma^2 = (1 + \frac{1}{4}K\bar{r}^2)^{-2}\left[d\bar{r}^2 + \bar{r}^2 d\Omega^2\right], \tag{36}$$

Combining with (27):

$$ds^2 = -dt^2 + S^2(t)\frac{d\bar{r}^2 + \bar{r}^2 d\Omega^2}{(1 + \frac{1}{4}K\bar{r}^2)^2}, \qquad (37)$$

And one more effort: it is convenient to leave only the sign of the K (which is a scale factor) as a physically relevant parameter: If $K \neq 0$ we can define $k = K/\|K\|$. If we define also $r^* = \|K\|^{1/2} r$ we would get:

$$ds^2 = -dt^2 + \frac{S^2(t)}{|K|}\left(\frac{dr^{*2}}{1 - r^{*2}} + r^{*2}(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad (38)$$

and defining a rescaled scaled function as well:

$$R(t) = S(t)/\|K\|^{1/2} \qquad if \qquad K \neq 0, \tag{39}$$

$$R(t) = S(t) \qquad if \qquad K = 0 \tag{40}$$

we get after dropping the stars, finally!:

$$ds^2 = -dt^2 + R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{41}$$

or in the $\bar{r}$ coordinate:

$$ds^2 = -dt^2 + R^2(t)\frac{d\bar{r}^2 + \bar{r}^2 d\Omega^2}{(1 + \frac{1}{4}K\bar{r}^2)^2}, \qquad (42)$$

where $k = +1, -1, 0$. (41) is called the Robertson-Walker metric.

The associated geometries

$k = +1$

We see in (41) that the coefficient of dr^2 becomes singular as $r \to 1$. We can go around with:

$$r = \sin \chi, \tag{43}$$

and,

$$dr = \cos \chi d\chi = (1 - r^2)^{1/2} d\chi, \tag{44}$$

and the 3-d part becomes:

$$d\sigma^2 = R_0^2 \left(d\chi^2 + \sin^2 \chi d\Omega^2 \right), \tag{45}$$

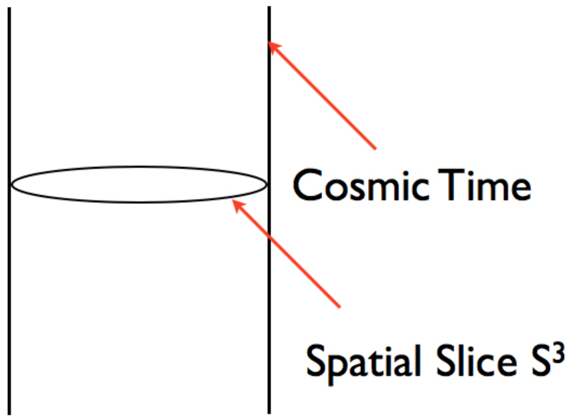
But now we can embed this 3-surface in a 4-dimensional Euclidean space (w, x, y, z) where

$$\left.\begin{aligned} w &= R_0 \cos\chi, \\ x &= R_0 \sin\chi \sin\theta \cos\phi, \\ y &= R_0 \sin\chi \sin\theta \sin\phi, \\ z &= R_0 \sin\chi \cos\theta. \end{aligned}\right\} \tag{46}$$

Now trivially:

$$d\sigma^2 = dw^2 + dx^2 + dy^2 + dz^2 = R_0^2 \left(d\chi^2 + \sin^2\chi d\Omega^2 \right), \tag{47}$$

which is in agreement with (44).



The topology with k=+1

$k = 0$ If we look at (41) at a given time $t = t_0$ the spatial part of the metric can become with the following coordinate choice:

$$\left.\begin{aligned} x &= R_0 \sin\theta \cos\phi, \\ y &= R_0 \sin\theta \sin\phi, \\ z &= R_0 \cos\theta. \end{aligned}\right\} \tag{48}$$

Then the metric becomes

$$d\sigma^2 = dx^2 + dy^2 + dz^2, \tag{49}$$

And the topology is the same as the $k = +1$ case.

$k = -1$

We can introduce a new coordinate $r = \sinh\chi$ and then,

$$dr = \cosh\chi d\chi = (1 + r^2)^{1/2} dr, \tag{50}$$

so

$$d\sigma^2 = R_0^2 \left(d\chi^2 + \sinh^2\chi d\Omega^2 \right), \tag{51}$$

Notice that now we can embed this 3-surface in a a flat Minkowski space:

$$d\sigma^2 = -dw^2 + dx^2 + dy^2 + dz^2, \tag{52}$$

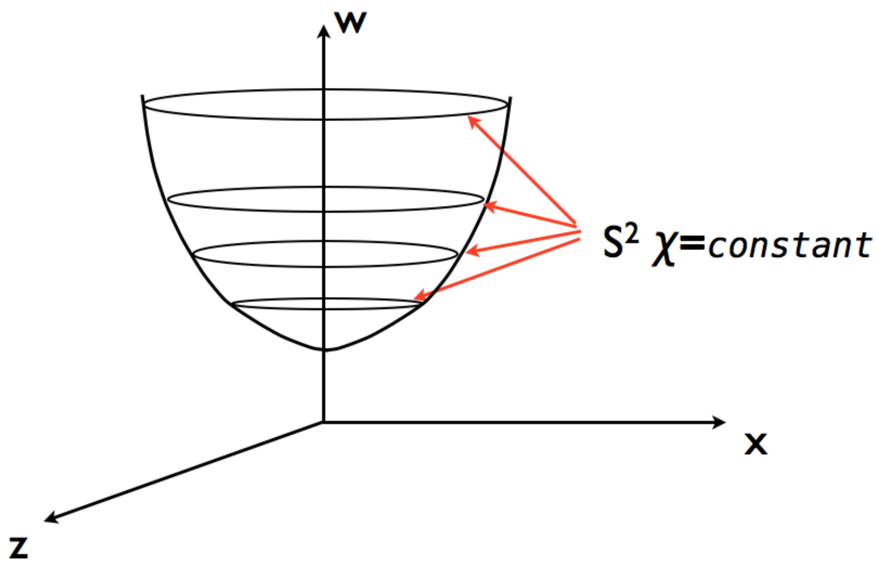
with the coordinates defined:

$$\left.\begin{aligned} w &= R_0 \cosh\chi, \\ x &= R_0 \sinh\chi \sin\theta \cos\phi, \\ y &= R_0 \sinh\chi \sin\theta \sin\phi, \\ z &= R_0 \sinh\chi \cos\theta. \end{aligned}\right\} \tag{53}$$

The equations imply that:

$$-w^2 + x^2 + y^2 + z^2 = R_0^2 \tag{54}$$

so the 3-surface is a three dimensional hyperboloid in four dimensional Minkowski space.



The topology with k=-1

The 2-surfaces in the picture are 2-spheres of surface area:

$$A_\chi = 4\pi R_0^2 \sinh^2 \chi$$

χ ranges from $0 \to \infty$ The 3-volume can become infinite.

Friedmann's equation

We will now work to develop relativistic cosmological models. We need:

- the FRW metric

$$ds^2 = -dt^2 + R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{55}$$

- Weyl's postulate

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu} \tag{56}$$

- Einstein's cosmological eqs:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{57}$$

Then using that in our comoving coordinate system $\vec{u} = (1, 0, 0, 0)$, the field equations become,

$$3\frac{\dot{R}^2 + k}{R^2} - \Lambda = 8\pi\rho, \qquad (58)$$

$$\frac{2R\ddot{R} + \dot{R}^2 + k}{R^2} - \Lambda = -8\pi p, \qquad (59)$$

Due to considerations of isotropy and homogeneity ρ and π can be only functions of time.

Differentiating (57) respect to time, multiply by $1/8\pi$ and add the result to (58) multiplied by $-3\dot{R}/8\pi R$

we get:

$$\dot{\rho} + 3p\frac{\dot{R}}{R} = -\frac{3}{8\pi}\frac{\dot{R}}{R}\left(\frac{3\dot{R}^2}{R^2} + \frac{3k}{R^2} - \Lambda\right) = -3\rho\frac{\dot{R}}{R}, \tag{60}$$

Multiplying by R^3 we can rewrite this:

$$\frac{d}{dt}\left(\rho R^3\right) + p\frac{d}{dt}\left(R^3\right) = 0, \tag{61}$$

But $R^3(t)$ is the volume of the fluid we are considering V. And ρV the total mass-energy in the volume V. But then we can rewrite (60),

$$dE + pdV = 0 \tag{62}$$

Notice that this is the law of conservation of energy. This is the result of satisfying the Bianchi identities. But if we consider experimental evidence, $p/\rho < 10^{-5}$, so we can assume $p = 0$. In that case (58) integrates immediately. First we need to multiply it by $\dot{R}$ throughout, and then identify that the left hand side is a total time derivative of the expression

$$R(\dot{R} + k) - \frac{1}{3}\Lambda R^3 = C \tag{63}$$

with C a constant of integration, which can be quickly identified using (57) as

$$C = \frac{8}{3}\pi R^3 \rho \tag{64}$$

This is twice the mass content of a spherical volume of a Euclidean universe of radius R and density ρ. But we can use (63) now to eliminate ρ in (57):

$$\dot{R}^2 = \frac{C}{R} + \frac{1}{3}\Lambda R^2 - k \qquad (65)$$

I will call R, which some authors call the scale factor, the ”radius of the Universe”. (64) is called the **Friedmann’s equation**. Compare (64) with (21)! The only difference is the cosmological constant term. Notice that in the Newtonian derivation we were looking at an expanding fluid with zero pressure.

Propagation of light

An observer sees light from a galaxy (which is receding). A radial null geodesic will have:

$$ds^2 = d\theta = d\phi = 0 \tag{66}$$

So using (41) we get

$$\frac{dt}{R(t)} = \pm \frac{dr}{(1 - kr^2)^{\frac{1}{2}}} \tag{67}$$

We can think of a time t_0 at which the observer is receiving light from a galaxy situated at a point with

$r = r_1$ and emitted at time t_1. Then

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = -\int_{r_1}^{0} \frac{dr}{(1-kr^2)^{\frac{1}{2}}} = f(r_1), \qquad (68)$$

where,

$$f(r_1) = \begin{cases} \sin^{-1} r_1 & \text{if } k = +1, \\ r_1 & \text{if } k = 0, \\ \sinh^{-1} r_1 & \text{if } k = -1. \end{cases}$$

Let's consider now two successive rays of light which are emitted by the galaxy at times t_1 and $t_1 + dt$

which are received by our observer at times t_0 and $t_0 + dt_0$. Following (67) we see that even if we calculate the left hand side for different intervals of time it will still be a function only of the position of the galaxy respect to the observer (i.e. r_1):

$$\int_{t_1+dt_1}^{t_0+dt_0} \frac{dt}{R(t)} = \int_{t_1}^{t_0} \frac{dt}{R(t)} = f(r_1), \qquad (69)$$

and then:

$$\int_{t_1+dt_1}^{t_0+dt_0} \frac{dt}{R(t)} - \int_{t_1}^{t_0} \frac{dt}{R(t)} = 0 =, \qquad (70)$$

$$\int_{t_0}^{t_0+dt_0} \frac{dt}{R(t)} - \int_{t_1}^{t_1+dt_1} \frac{dt}{R(t)} \qquad (71)$$

If $R(t)$ doesn't vary much over dt_1 and dt_0 we get:

$$\frac{dt_0}{R(t_0)} = \frac{dt_1}{R(t_1)}. \tag{72}$$

The galaxies move along world lines on which the coordinates r, θ, ϕ are constant.
Consequently, $ds^2 = dt^2$ which implies that t measures the proper time along the fluid particles (galaxies) world lines. dt_1 and dt_0 are the proper time intervals between the two light rays emitted by the galaxy and perceived by the observer. This means according to (71) that the time interval measured by the observer is $R(t_0)/R(t_1)$ times the time interval

measured by someone at the emitter galaxy. The universe is expanding, so $t_0 > t_1$ which implies $R(t_0) > R(t_1)$. This implies that the observer O will perceive a redshift z given by

$$1 + z = \frac{\nu_1}{\nu_0} = \frac{R(t_0)}{R(t_1)}. \tag{73}$$

where ν_1 and ν_0 are the frequencies measured by the emitter and the receiver. This the cosmological redshift. For short time differences $t_0 = t_1 + dt$

and so (72) will become:

$$1 + z = \frac{\nu_1}{\nu_0} = \frac{R(t_0)}{R(t_0 - dt)} \simeq \frac{R(t_0)}{R(t_0) - \dot{R}_{t_0} dt} \tag{74}$$

$$\approx 1 + \frac{\dot{R}(t_0)}{R(t_0)} dt \tag{75}$$

If we integrate,

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_{t_1}^{t_1 + dt} \frac{dt}{R(t)} \approx \frac{dt}{R(t_1)} = \tag{76}$$

$$\frac{dt}{R(t_0 - dt)} \approx \frac{dt}{R(t_0)} \tag{77}$$

In the case of small r, we get from $f(r)$ as defined in (67) and the formula below,

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = f(r_1) \approx r_1, \tag{78}$$

So we can write now:

$$\frac{dt}{R(t_0)} \approx r_1 \tag{79}$$

And from (74)

$$z \approx \dot{R}(t_0) r_1 \tag{80}$$

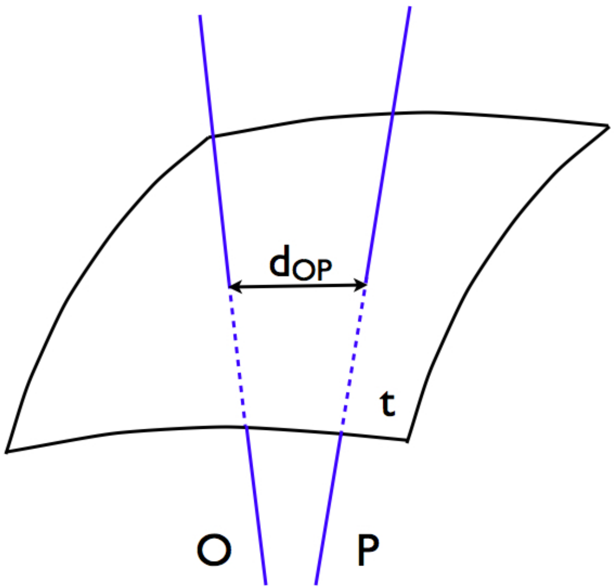
Distance in Cosmology

An easy way to measure distance is to use the world time and measure the distance (absolute at the same time on a given slice) between particles by just measuring the proper distance along a geodesic line. We set $dt = d\theta = d\phi = 0$ in (41). Then,

$$d_A = R(t) \int_0^{r_1} \frac{dr}{(1 - kr^2)^{1/2}} \qquad (81)$$

But we would need to know R_1.

Because what we know it is the apparent luminosity of the particle, i.e. the galaxy or nebula, we may try it. If E is the total luminous energy radiated per unit of time by the galaxy, and I the intensity of the radiation measured by the observer per unit of area and time.



Cosmological distance

We can define the distance as $(E/4\pi I)^{1/2}$. There is an issue with the time though. We need to account for the Doppler shift of the light (a result of the expanding universe!) so the number of photons will be reduced but also their energy (frequency) will be reduced. So the redshift factor enters twice! and we have:

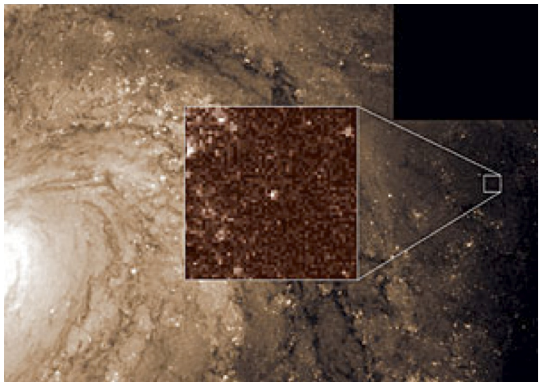
$$d_L^2 = \frac{E}{4\pi I(1+z)^2} \tag{82}$$

This is called the luminosity distance. This formula is just an approximation. The main problem is that we can not measure in general E.

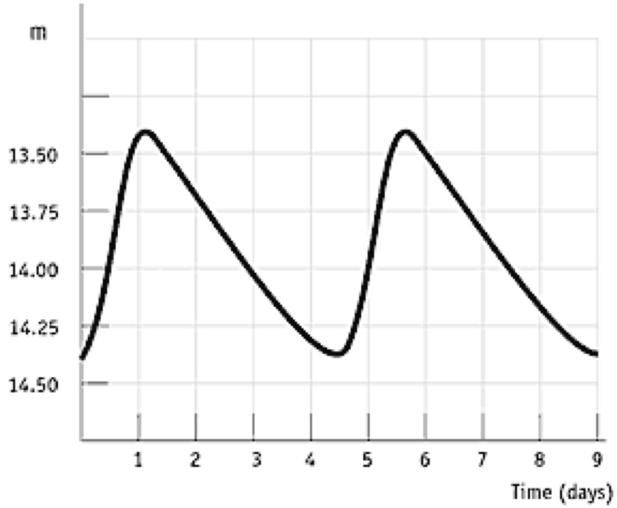
A digression on measuring distance to the stars

In 1912 Henrietta Leavitt found the following relationship between absolute magnitude and the period of variable stars called Cepheids:

$$M = -2.78 \log P - 1.35 \tag{83}$$



Cepheid stars in M100



Light curve for a cepheid.

The apparent magnitude is related to the absolute magnitude (or energy emitted) and can be related to the energy received (which is itself related to the distance to the source by (74).

The relationship between the apparent magnitude and the energy received is given by:

$$m = constant - 0.4 \log_{10} E_R. \tag{84}$$

In a similar manner distances can be inferred from luminosity measurements of nearby galaxies,

$$L = 4\pi d^2 F$$

where d is the distance to the galaxy and F is the flux we measure. From this we can define the luminosity distance

$$d_L = \left(\frac{L}{4\pi F}\right)^{1/2}$$

What is the relationship between the luminosity distance and the cosmological scales? Suppose that our source gives only one type of photons of frequency ν_e at time t_e What is the flux at a later time t_0? In an interval δt_e the source emits:

$$N = L\delta t_e / h\nu_e$$

To find the flux we need to know the area of the sphere that these photons occupy at the time we observe them.

Integrating over the spherical angles using (54) we get:

$$A = 4\pi {R_0}^2 r^2$$

but the photons have redshifted by $1+z = R_0/R(t_e)$ to a frequency ν_0:

$$h\nu_0 = h\nu_e/(1+z)$$

They would arrive during a time δt_0

$$\delta t_0 = \delta t_e(1+z)$$

The flux of light at time t_0 is then $Nh\nu_0/(A\delta t_0)$, and then,

$$F = L/A(1+z)^2$$

Then the luminosity distance d_L is,

$$d_L = R_0 r(1+z)$$

To get the final formula we need to have the value of r as a function of the redshift z. All we need to do is to set $ds = 0$ in (54) which gives us

$$\frac{dr}{(1-kr^2)^{1/2}} = -\frac{dt}{R(t)}$$

Hubble's law

If light coming from P_1 at time t_1 is observed "now" by an observer at O at time t_0 where $t_1 < t_0$ the light will be extended over a sphere with centre at P_0 where $t = t_0$ and $r = r_1$ and passing through

O_0 with $t = t_0$ and $r = 0$. The surface area of the sphere centered at O_0 is the same as the one centered at P_0. We have to remember that the 3-sphere is homogeneous. From (41) the line element of the 3-sphere is:

$$ds^2 = [R(t_0)r_1]^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (85)$$

The integration of $d\Omega^2$ will give the sphere with surface $4\pi R^2(t_0)r_1^2$ and consequently the observed intensity for the galaxy's light emitted at P_1 is

$$I = \frac{E}{4\pi r_1^2 R^2(t_0)(1+z)^2}, \qquad (86)$$

Comparing with (81) we get:

$$d_L = r_1 R(t_0). \tag{87}$$

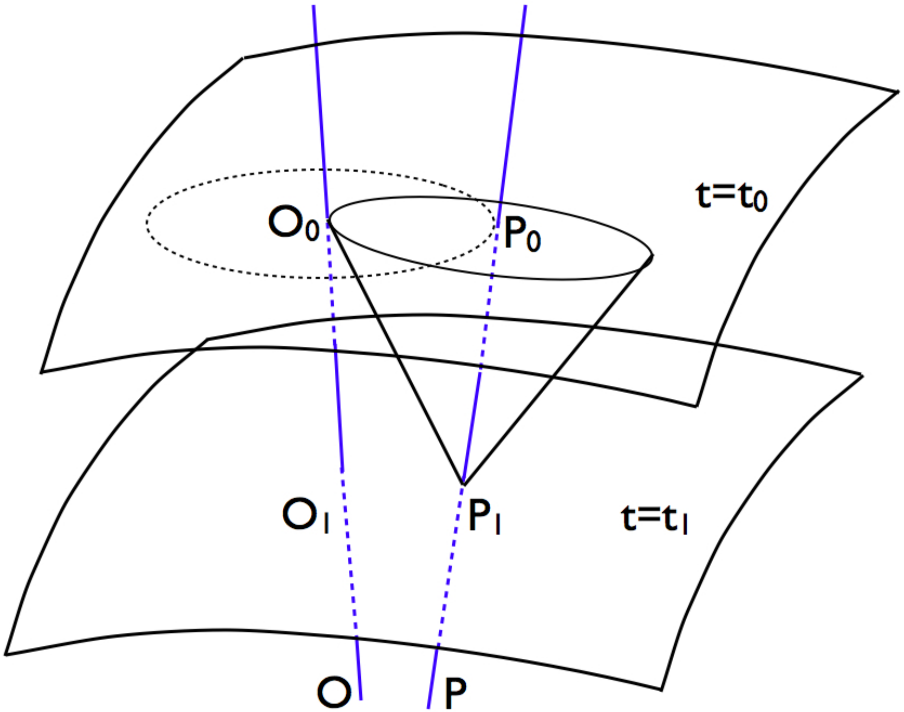
If we define the Hubble parameter

$$H(t) = \frac{\dot{R}(t)}{R(t)} \tag{88}$$

the we have

$$z \approx H(t_0) d_L, \tag{89}$$

H_0 is the value of the Hubble parameter at the current epoch an is called the Hubble constant.



The sphere of light from P_1 at O_0

According to the WMAP results the most current value of the Hubble's constant is 73.5 ± 3.2 km/sec/Mpc. This is $H_0 = 2.3 \times 10^{-18}$ 1/sec.
The Hubble time is $T = 1/H_0 = 4.35 \times 10^{17}$ sec.
The velocity of recession of galaxies as measured by their redshift is proportional to its distance. The deceleration parameter q is

$$q(t) = -\frac{R\ddot{R}}{\dot{R}^2} \qquad (90)$$

q measures the rate at which the expansion of the universe is slowing down. Current estimates get a negative value, meaning the universe expansion is not subduing but increasing. From (74) we can include second order effects into account and find that:

$$d_L = zT_0[1 - \frac{1}{2}(1 + q_0)z + \cdots]. \qquad (91)$$

(88) is fine for nearby galaxies. But beyond 18th magnitude (90) has to be used. Notice that this latter one is a function of q_0.

Differentiating (64)

$$2\dot{R}\ddot{R} = -\frac{C}{R^2}\dot{R} + \frac{2}{3}\Lambda R\dot{R}, \qquad (92)$$

and multiplying by $-R/2\dot{R}^3$ we get,

$$-\frac{R\ddot{R}}{\dot{R}^2} = \frac{C}{2R\dot{R}^2} - \frac{1}{3}\Lambda\frac{R^2}{\dot{R}^2}. \tag{93}$$

Then from (89), (63) and (87) we get:

$$q = \left(\frac{4}{3}\pi\rho - \frac{1}{3}\Lambda\right)/H^2 \tag{94}$$

Another important observable is N, the number of galaxies in a given volume. The volume is given by:

$$V = 4\pi R^3(t_0)\int_0^{r_1}\frac{r^2dr}{(1-kr^2)^{1/2}}. \tag{95}$$

The number of galaxies in this volume is

$$N = Vn(t_0). \tag{96}$$

Is this number constant? We need a theory of galactic evolution. H, q, ρ and N play a crucial role in determining different models and possible evolutions for our universe.

We can go back to

$$\frac{dr}{(1-kr^2)^{1/2}} = -\frac{dt}{R(t)}$$

and we see that we can now put

$$\frac{dr}{(1-kr^2)^{1/2}} = -\frac{dt}{R(t)} = \frac{dz}{R_0 H(z)},$$

after using that $H = \frac{\dot{R}(t)}{R(t)}$ and also eq (72)

$$1 + z = \frac{R(t_0)}{R(t_1)}$$

And integrating assuming small r and z and working only to first order beyond the Euclidean relations:

$$d_L = R_0 r(1+z) = \left(\frac{z}{H_0}\right)\left[1 + \left(1 + \frac{1}{2}\frac{\dot{H}_0}{{H_0}^2}\right)z\right] + \cdots$$

If we can measure the luminosity distances and redshifts of a number of objects, then we can measure $\dot{H}_0$.

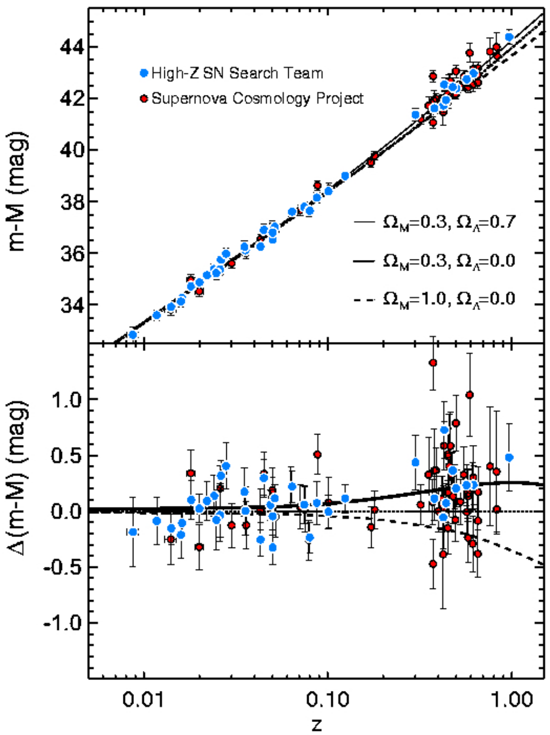
The Universe is accelerating

The way this is observed is by doing a plot of the luminosity distance against redshift for Type Ia supernovae.

These occur occur in binary systems in which one of the stars is a white dwarf while the other can vary from a giant star to an even smaller white dwarf.

This category of supernovae produces a consistent peak luminosity because of the uniform mass of white dwarfs that explode via the accretion mechanism (carbon-oxygen white dwarfs with a low rate of rotation are limited to below 1.38 solar masses).

The stability of this value allows these explosions to be used as standard candles to measure the distance to their host galaxies because the visual magnitude of the supernovae depends primarily on the distance.



L_d vs z for Type Ia supernovae

Cosmological models

Flat case $k = 0$

In this case we get:

$$\dot{R}^2 = C/R + \frac{1}{3}\Lambda R^2. \tag{97}$$

We assume $\Lambda > 0$ and introduce a new variable u:

$$u = \frac{2\Lambda}{3C} R^3. \tag{98}$$

Differentiating,

$$\dot{u} = \frac{2\Lambda}{C} R^2 \dot{R}, \tag{99}$$

and substituting in (97)

$$\begin{aligned}
\dot{u}^2 &= \frac{4\Lambda^2}{C^2} R^4 \left(\frac{C}{R} + \frac{1}{3} \Lambda R^2 \right) && (100)\\
&= \frac{4\Lambda^2}{C} R^3 + \frac{4\Lambda^3}{3C^2} R^6 && (101)\\
&= 6\Lambda u + 3\Lambda u^2 && (102)\\
&= 3\Lambda(2u + u^2). && (103)
\end{aligned}$$

or,

$$\dot{u} = (3\Lambda)^{\frac{1}{2}}(2u + u^2)^{\frac{1}{2}}. \qquad (104)$$

This equation can be integrated by parts...
Assuming $R = 0$ when $t = 0$, then $u = 0$ and we have,

$$\int_0^u \frac{du}{(2u + u^2)^{\frac{1}{2}}} = \int_0^t (3\Lambda)^{1/2} dt = (3\Lambda)^{1/2} t. \qquad (105)$$

Completing squares and making $v = u + 1$ and $\cosh w = v$

$$\int_0^u \frac{du}{[(u+1)^2 - 1]^{\frac{1}{2}}} = \int_1^v \frac{\sinh w dw}{(\cosh^2 w - 1)^{1/2}} \tag{106}$$

$$= \int_0^w dw = w. \tag{107}$$

And going back to R,

$$R^3 = \frac{3C}{2\Lambda}\left[\cosh(3\Lambda)^{1/2} t - 1\right]. \tag{108}$$

If $\Lambda < 0$ we can introduce:

$$u = -\frac{2\Lambda}{3C} R^3 \tag{109}$$

and then we can get,

$$R^3 = \frac{3C}{2(-\Lambda)} \left\{ 1 - \cosh \left[3 \left(-\Lambda \right) \right]^{\frac{1}{2}} t \right\}. \tag{110}$$

If $\Lambda = 0$

$$\dot{R} = \left(\frac{C}{R} \right)^{1/2}. \tag{111}$$

Direct integration gives,

$$R = \left(\frac{9}{4}Ct^2\right)^{\frac{1}{3}}. \tag{112}$$

This is the **Einstein-de Sitter** model. The Hubble parameter is:

$$H(t) = \dot{R}/R = 2/(3t). \tag{113}$$

The deceleration parameter is:

$$q(t) = -R\ddot{R}/\dot{R}^2 = \frac{1}{2}. \tag{114}$$

At the beginning of the expanding universe, R is small and C/R dominates. So for small t

$$\dot{R}^2 \sim C/R, \tag{115}$$

integrating,

$$R \sim \left(\frac{9}{4}Ct^2\right)^{\frac{1}{3}}. \tag{116}$$

In early stages all models regardless of the value of Λ expand like $t^{2/3}$

Models with vanishing cosmological constant

$$\dot{R}^2 = C/R - k, \tag{117}$$

We consider two cases $k = +1$ and $k = -1$ **k=+1** (116) becomes

$$\dot{R}^2 = C/R - 1, \tag{118}$$

We define:

$$u^2 = R/C, \tag{119}$$

And then $2u\dot{u} = \dot{R}/C$, and substituting in (116):

$$\dot{u}^2 = \frac{\dot{R}^2}{4C^2u^2} = \frac{1}{4C^2u^2}\left(\frac{C}{R} - 1\right) = \frac{1}{4C^2u^2}\left(\frac{1}{u^2} - 1\right) \tag{120}$$

The equation is separable if we take positive square roots,

$$2\int_0^u \frac{u^2}{(1-u^2)^{\frac{1}{2}}}du = \frac{1}{C}\int_0^t dt = \frac{t}{C} \qquad (121)$$

We can evaluate the $u-$integral, we make $u = sin\theta$.

$$2\int_0^u \frac{u^2}{(1-u^2)^{\frac{1}{2}}}du = 2\int_0^\theta \frac{sin^2\theta cos\theta d\theta}{(1-sin^2\theta)^{1/2}} = \qquad (122)$$

$$sin^{-1}u - u(1-u^2)^{1/2} \qquad (123)$$

which back to R yields,

$$C[\sin^{-1}(R/C)^{1/2} - (R/C)^{1/2}(1 - R/C)^{1/2}] = t. \tag{124}$$

In the case **k=-1** we get...

$$C[(R/C)^{1/2}(1 + R/C)^{1/2} - \sinh^{-1}(R/C)^{1/2}] = t. \tag{125}$$

The case $\lambda = 0$, $k = 0$ is the Einstein-de Sitter model of eq (115) The Hubble and deceleration parameters are,

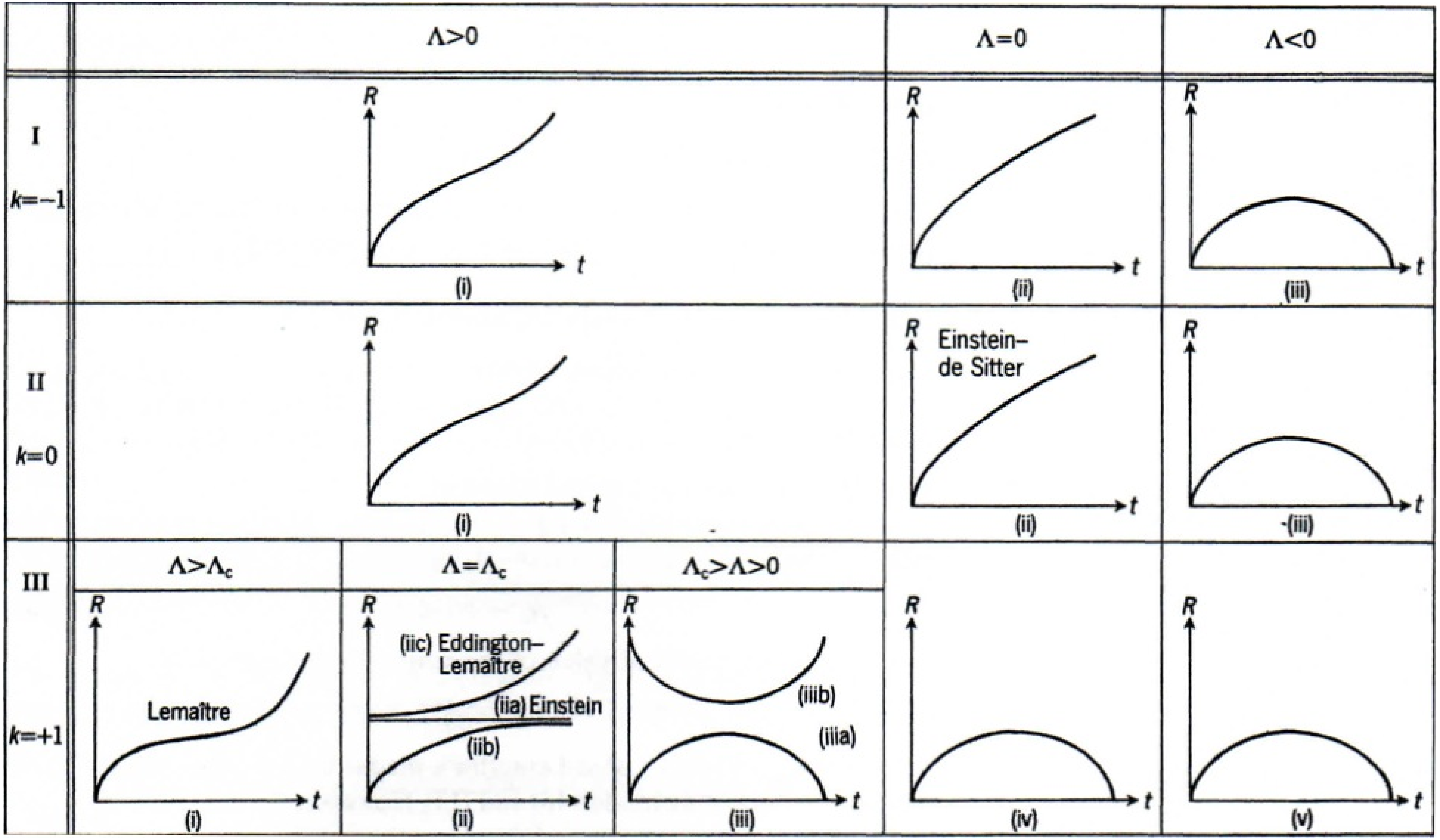
$$H = C^{-1}(R/C)^{-3/2}(1 - R/C)^{1/2} \tag{126}$$

$$q = \frac{1}{2}(1 - R/C)^{-1} \tag{127}$$

where R is a function of (t) implicitly from (123).
We can write (117) as $\dot{R}^2 = G(R)$, and then we get

$$G(R) = C/R - k, \tag{128}$$

We can see that $k = +1$ has a local minimum, while the other models grow without bound.
If $k = -1$, and $t \to \infty$ we'll get $\dot{R}^2 \sim 1$, and then $R \sim t$.



Friedmann models

Friedman-Robertson Walker universes

Using (54) and assuming that the matter content is a perfect fluid, we can look at $T^{\mu\nu}{}_{;\nu} = 0$. The only non trivial component is $\mu = 0$ and we get:

$$\frac{d}{dt}\left(\rho R^3\right) = -p\frac{d}{dt}\left(R^3\right) \tag{129}$$

where $R(t)$ is the cosmological expansion factor. R^3 is proportional to the volume of the fluid and then the left hand of (128) is the rate of change of energy, and the right hand is the work it does as it expands $(-pdV)$.

In a matter dominated universe we have $p = 0$ and then

$$\frac{d}{dt}\left(\rho R^3\right) = 0 \tag{130}$$

In a radiation dominated era, $p = \frac{1}{3}\rho$

$$\frac{d}{dt}\left(\rho R^3\right) = -\frac{1}{3}\rho\frac{d}{dt}\left(R^3\right) \tag{131}$$

or

$$\frac{d}{dt}\left(\rho R^3\right) = 0 \tag{132}$$

In Einstein's els the only two components non zero are G_{tt} and G_{rr}, so only one component survives (due to Bianchi's identities):

$$G_{tt} = 3(\frac{\dot{R}}{R})^2 + 3k/R^2 \tag{133}$$

So besides (129) or (131) we have Einstein's eq with a cosmological constant Λ

$$G_{tt} + \Lambda g_{tt} = 8\pi T_{tt} \tag{134}$$

We can think then of Λ as the energy density and pressure of a fluid

$$\rho_\Lambda = \Lambda/8\pi, \quad p_\Lambda = -\rho_\Lambda \tag{135}$$

ρ_Λ is called the dark energy.

Then Einstein's eqs can be written,

$$\frac{1}{2}\dot{R}^2 = -\frac{1}{2}k + \frac{4}{3}\pi R^2(\rho_m + \rho_\Lambda) \qquad (136)$$

From (128) and the time derivative of (135) we get

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}(\rho + 3p), \qquad (137)$$

where ρ and p are the total energy and density pressure for both matter and dark energy.