Lesson 11 A
B-H Thermodynamics

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The Penrose process and Black Hole Thermodynamics[a]

[a] This lesson follows mostly Chapter 6 from S. Carroll's book

We will study geodesics in the Kerr metric. Remembering that we have $T = \partial/\partial t$ and $R = \partial/\partial \phi$ as Killing vectors we can study the conserved quantities associated with them. The four momentum:

$$p^\mu = m \frac{dx^\mu}{d\tau} \tag{1}$$

where m is a particle's rest mass. We can construct the energy:

$$E = -T_\mu p^\mu = m \left(1 - \frac{2GMr}{\rho^2}\right) \frac{dt}{d\tau} + \frac{2mGMar}{\rho^2} \sin^2\theta \frac{d\phi}{d\tau} \tag{2}$$

and the angular momentum associated with R. With $L = R_\mu p^\mu$

$$L = -\frac{2mGMar}{\rho^2} \sin^2\theta \frac{dt}{d\tau} + \frac{m(r^2+a^2)^2 - m\Delta a^2 \sin^2\theta)}{\rho^2} \sin^2\theta \frac{d\phi}{d\tau} \tag{3}$$

The minus sign in eq (2) reflects the fact that at infinity both T^μ and p^μ are timelike so their inner product is negative, but we need the energy to be positive. But inside the ergosphere T^μ becomes spacelike. So we can think of particles for which

$$E = -T_\mu p^\mu < 0 \tag{4}$$

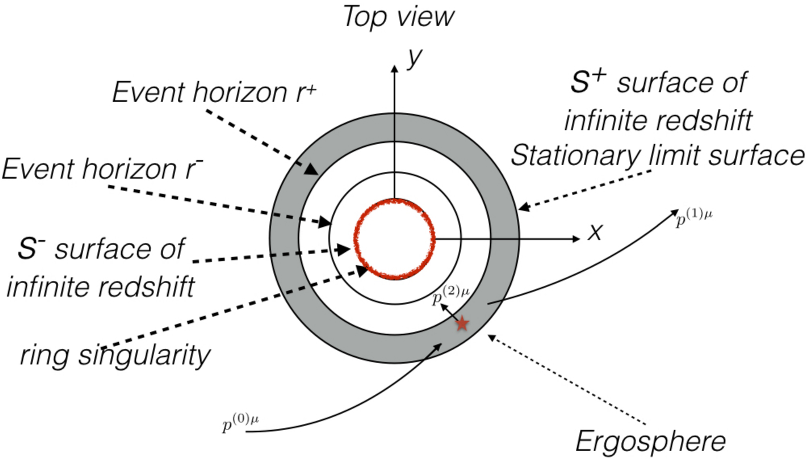
The Penrose process rests on a simple idea: from outside the ergosphere we can have an observer with a massive rock. The momentum of the system observer plus rock is $p^{(0)\mu}$. The energy $E^{(0)} = -T_\mu p^{(0)\mu}$ is positive. The observer and rock momenta are $p^{(1)\mu}$ and $p^{(2)\mu}$ respectively

$$p^{(0)\mu} = p^{(1)\mu} + p^{(2)\mu} \tag{5}$$

contracting with the Killing vector their energy is

$$E^{(0)} = E^{(1)} + E^{(2)} \tag{6}$$

If the rock is thrown beyond the stationary surface into the inner E-H of the B-H with $E^{(2)} < 0$



The Penrose process: an object breaks down in two as it falls into the ergosphere, within the stationary limit but outside the outer event horizon.

At the end

$$E^{(1)} > E^{(0)} \tag{7}$$

Killing Horizons

A null hypersurface at which a Killing vector field χ^μ becomes null is called a **Killing Horizon** (K-H). How does it relate to event horizons (E-H)?

1. Every E-H Σ in a stationary, asymptotically flat S-T is a K-H for some Killing vector field χ^μ.
2. If the S-T is static, the χ^μ will be the Killing vector field $T^\mu = (\partial/\partial t)^\mu$ representing time translations at infinity.
3. If the S-T is stationary $\chi^\mu = T^\mu + \Omega_H R^\mu$ where Ω_H is some constant and T^μ is as in 2) and $R^\mu = (\partial/\partial\phi)^\mu$.

Note: there could exist K-H even when no E-H exist: in Minkowski there a re no E-H. A Killing vector $\chi = x\partial_t + t\partial_x$ (it generates boosts in the x-direction) has norm $\chi^\mu\chi_\mu = -x^2 + t^2$ which goes null at $x = \pm t$.

There is energy gained: the Penrose process extracts energy from the rotating B-H by decreasing its angular momentum: the object has to be thrown against the hole's rotation. For Kerr B-H an E-H is a K-H for $\chi^\mu = T^\mu + \Omega_H R^\mu$ where $\Omega_H = \frac{a}{r^2{}_+ + a^2}$ is the angular velocity of the horizon (r_+ is defined in eq 115, slide 54). The angular velocity can be obtained by studying the trajectory of a photon emitted in the ϕ direction in the equatorial plane ($\theta = \pi/2$). In this case $ds^2 = 0 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2$ which yields $\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}$. AT the stationary limit $g_{tt} = 0$ and we can define it a the E-H by evaluating it at r_+.

$$\Omega_H = \frac{a}{r^2{}_+ + a^2} \tag{8}$$

χ^μ becomes null at the outer E-H by construction. The thrown object will be crossing the E-H forward in time, which implies $p^{(2)\mu}\chi_\mu < 0$. Using (2) and (3) this latter condition becomes $p^{(2)\mu}T_\mu + \Omega_H p^{(2)\mu}R_\mu = -E^{(2)} + \Omega_H L^{(2)}$. from this we get:

$$L^{(2)} < \frac{E^{(2)}}{\Omega_H} \tag{9}$$

But we required $E^{(2)}$ to be negative (and Ω_H is positive) which means then that $L^{(2)}$ is negative and the particle must have negative angular momentum, i.e. move against the hole's rotation. But the mass and angular momentum of the hole will change proportionally i.e. $\delta M = E^{(2)}$ and $\delta J = L^{(2)}$, where $J = Ma$ is the angular momentum of the B-H. Then (9) gives us a limit on how much can we decrease the angular momentum

$$\delta J < \frac{\delta M}{\Omega_H}. \tag{10}$$

In a "perfect" process this can become and identity. But M can not be decreased beyond a fundamental limit.

The area Theorem

The area of the outer E-H is located at

$$r_+ = M + \sqrt{M^2 - a^2} \tag{11}$$

and can be calculated over θ, ϕ with $dt = dr = 0$ in the Kerr metric

$$ds^2(r = r_+) = ({r^2}_+ + a^2 cos^2\theta)d\theta^2 + \left[\frac{({r^2}_+ + a^2)^2 sin^2\theta}{{r^2}_+ + a^2\cos^2\theta}\right] d\phi^2. \tag{12}$$

The horizon area is given by the integral of the induced volume element $ds^2 = \gamma_{ij}dx^i dx^j$ in (12) as

$$A = \int \sqrt{|\gamma|} d\theta d\phi. \tag{13}$$

The determinant $|\gamma| = ({r^2}_+ + a^2)^2 sin^2\theta$, so $A = 4\pi({r^2}_+ + a^2)$

Let's define now the **irreducible mass** of the B-H ${M^2}_{irr} = A$. The reason for the name will become clear soon.

$$
\begin{aligned}
{M^2}_{irr} = A = 4\pi({r^2}_+ + a^2) = \\
= 8\pi\left(M^2 + \sqrt{M^4 - J^2}\right)
\end{aligned}
\tag{14}
$$

where we use eq(11) and $J = Ma$. Differentiating we can calculate a variation in the irreducible mass:

$$
\delta M_{irr} = \frac{8\pi a}{\sqrt{M^2 - a^2}}\left(\frac{\delta M}{\Omega_H} - \delta J\right). \tag{15}
$$

And we see from (10) that we get

$$
\delta M_{irr} > 0. \tag{16}
$$

the maximum of energy we can extract from a Kerr B-H is

$$M - M_{irr} = M - \left[8\pi\left(M^2 + \sqrt{M^4 - J^2}\right)\right]^{1/2} \tag{17}$$

The result will be a Schwarzschild B-H of mass M_{irr}. From (14) we can calculate δA and we get

$$\delta M = \kappa\delta A + \Omega_H \delta J \tag{18}$$

where

$$\kappa = \frac{\sqrt{M^2 - a^2}}{16\pi M(M + \sqrt{M^2 - a^2})} \tag{19}$$

κ is called the surface gravity. Eq (18) is highly reminiscent of

$$dE = TdS - pdV \tag{20}$$

from thermodynamics. We can think of $\Omega_H \delta J$ in (18) as work done on the hole by throwing "mass" at it.

This would quickly lead to associate

$$
\begin{aligned}
E &\leftrightarrow M \\
S &\leftrightarrow 8\pi A \\
T &\leftrightarrow \alpha\kappa
\end{aligned}
\tag{21}
$$

We can enunciate then the Laws of B-H thermodynamics:

1. Zero law: Stationary B-Hs have constant surface gravity on the entire horizon (equivalent to the temperature being constant in a system in thermal equilibrium.
2. First law will be (18) which is equivalent to (20).
3. Second law: the area of a horizon never decreases (equivalent to the entropy).
4. a Third law doesn't quite exist for B-H due to the fact that $\kappa = 0$ -the equivalent to $T = 0$- corresponds to extreme B-Hs while in classical thermodynamics $T - 0$ cannot be reached.

The correspondence (21) has some caveats. But Hawking showed that quantum fields in a curved background allow the B-H to radiate at a $T \propto \kappa$. With this result it is completely appropriate to interpret then A as proportional to the entropy of the B-H. Bekenstein proposed a generalized 2nd law

$$\delta\left(S + \frac{A}{4G}\right) \geqslant 0 \tag{22}$$

This latter result in units where $\hbar = c = k = 1$. If we want to cast this result in terms of information theory (entropy proportional to the number of quantum states accessible to the B-H) there seems to be a contradiction with the fact that only mass and spin (and charge) characterize the B-H. This indicates a need for a quantum theory of gravity, which we don't have yet.