Introduction to General Relativity 2025
Lesson 6:

Mario C Díaz

1 The principle of equivalence

There are different versions, stronger or weaker, of the principle:

- The motion of a gravitational test particle in a gravitational field is independent of its mass and it composition.
- The gravitational field is coupled to everything.
- There are no local experiments which can distinguish non-rotating free fall in a gravitational field from uniform motion in space in the absence of a gravitational field.
- A frame linearly accelerated relative to an inertial frame in special relativity is locally identical to a frame at rest in a gravitational field.

Notice that we can formulate this mathematically in the following language: A test particle in Minkowski moves according to:

$$\frac{d^2x^a}{d\tau^2} = 0. \tag{1}$$

In a noninertial system of reference:

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a{}_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0. \tag{2}$$

Notice that if want to regard $\Gamma^a{}_{bc}$ as force terms, then g_{ab} has to be seen as potentials.

We need to generalize these ideas to build a relativistic theory of gravitation.

2 The principle of general covariance

Einstein propose then the following two principles to build a theory of gravitation consistent with relativity:

- *Principle of General Relativity*
  All observers are equivalent. In special relativity we have preferred systems, Minkowski coordinates. In a general curved space time we don't have a preferred coordinate system. (although there are symmetries).

- *Principle of General Covariance*
  The equations of physics should have tensorial form. What this means is that the theory should be invariant under coordinate transformations.

3 The principle of minimal gravitational coupling

This is a simplicity principle when making the transition to general relativity from special relativity. i.e. if we have the conservation law:

$$\partial_b T^{ab} = 0 \tag{3}$$

The simplest generalization is:

$$\nabla_b T^{ab} = 0 \tag{4}$$

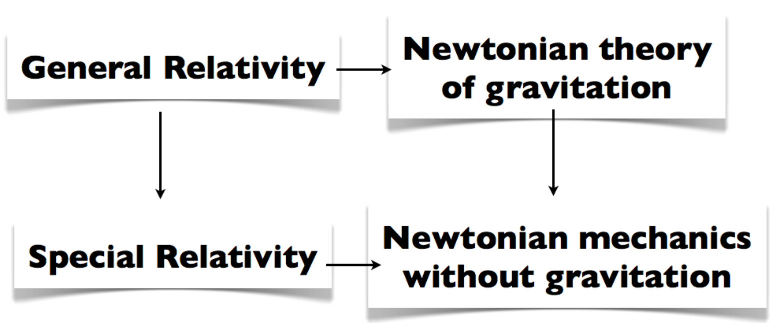
We could have taken:

$$\nabla_b T^{ab} + g^{be} R^a{}_{bcd} \nabla_e T^{cd} = 0 \tag{5}$$

but the Principle would stay: "No terms explicitly containing the curvature tensor should be added when making the transition."

4 The correspondence principle

The correspondence principle states simply that any theory of General Relativity should contain in the appropriate limit other theories that have stand the test of time. In the weak field limit, far from the sources, when speeds of the bodies involved are low, GR should reproduce Newton's theory of gravitation. In the absence of masses GR should become Special Relativity.



5 The full field equations

Let's revisit the Einstein's field equations:
The information about all the fields and forms of energy acting or present in a region of space time can be encoded in the energy-momentum tensor T^{ab}. After all the equivalence of mass and energy suggest that all forms of energy act as sources for the gravitational field.

If we assume that T^{ab} is the source of the field equations we know that:

$$\partial_b T^{ab} = 0 \tag{6}$$

This can be generalized to:

$$\nabla_b T^{ab} = 0 \tag{7}$$

But we also know that the Einstein tensor satisfies the Bianchi identities:

$$\nabla_b G^{ab} \equiv 0 \tag{8}$$

It was precisely this fact what suggested that the two tensors are proportional to one another. Thus we have:

$$G^{ab} = \kappa T^{ab} \tag{9}$$

In no relativistic units we will see that κ following the correspondence principle:

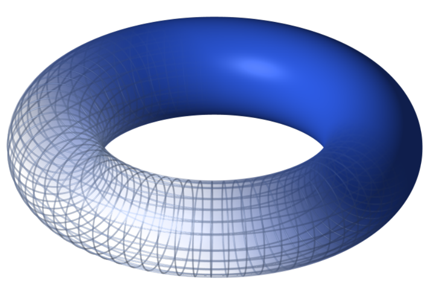
$$\kappa = 8\pi G/c^4 \tag{10}$$

So the full GR equations are:

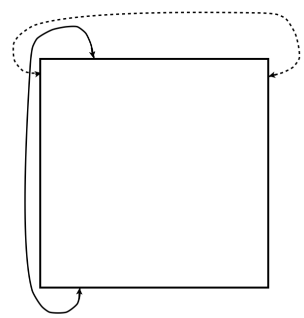
$$G^{ab} = \frac{8\pi G}{c^4} T^{ab} \tag{11}$$

6 Another look at metrics and curvature

If we compare a straight line and a circle we would say that the first one has no curvature but that the second one is curved, but after our definition of the Riemann tensor this can not be true. The Riemann tensor is identically zero in one dimension. Our intuitive definition of curvature is always thinking about an embedding in Euclidean space. Let's discuss the following example: A torus is an object in two dimensions where the Riemann tensor has only one independent component.



A torus can be considered as a square region of the plane with opposite sides identified.



A valid metric for the torus could be the following one with $0 < u, v < 2\pi$

$$ds^2 = (c + a\cos v)^2 du^2 + a^2 \sin^2 dv^2 \tag{12}$$

The following transformation

$$x = (c + a\cos v)\cos u \tag{13}$$
$$y = (c + a\cos v)\sin u \tag{14}$$

could change it into $ds^2 = dx^2 + dy^2$.

On the other hand let's look at the sphere $\mathbf{S}^2$:

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{15}$$

The non=zero connection coefficients for (39) are:

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \tag{16}$$
$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta \tag{17}$$

And the components of the Riemann tensor:

$$R^\theta_{\phi\theta\phi} = \partial_\theta \Gamma^\theta_{\phi\phi} - \partial_\phi \Gamma^\theta_{\theta\phi} + \Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{\phi\phi} - \Gamma^\theta_{\phi\lambda}\Gamma^\lambda_{\theta\phi} \tag{18}$$
$$= (\sin^2\theta - \cos^2\theta) - 0 + 0 - (-\sin\theta\cos\theta)(\cot\theta) \tag{19}$$
$$= \sin^2\theta \tag{20}$$

lowering the index we get:

$$R_{\theta\phi\theta\phi} = g_{\theta\lambda} R^\lambda \phi\theta\phi \tag{21}$$
$$= g_{\theta\theta} R^\theta \phi\theta\phi \tag{22}$$
$$= a^2 \sin^2\theta \tag{23}$$

Computing the Ricci tensor $R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$:

$$R_{\theta\theta} = g^{\phi\phi} R_{\phi\theta\phi\theta} = 1 \tag{24}$$

$$R_{\theta\phi} = R_{\phi\theta} = 0 \tag{25}$$

$$R_{\phi\phi} = g^{\theta\theta} R_{\theta\phi\theta\phi} = \sin^2\theta \tag{26}$$

And the Ricci scalar is:

$$R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{2}{a^2}. \tag{27}$$

7 Physics in slightly curved space-times

Let's review some concepts from Special Relativity that are crucial to understand General Relativity. Space time is the arena of Special Relativity. A single point in this S-T is a set $\{t, x^i\}$ where in general $i = 1, 2, 3$. This is what we call an event. A line giving the position of a particle as time evolves is called the world line. Let's summarize then the main concepts that takes us from differential geometry to a theory of gravity.

- Spacetime understood as the set of all events, is a four dimensional manifold endowed with a metric.
- The metric is measurable by rulers and clocks. The distance along a ruler between two adjacent points is $|d\vec{x} \cdot d\vec{x}]^{1/2}$ and the time measure by two clocks for events that are close in time is $| - d\vec{x} \cdot d\vec{x}]^{1/2}$.
- The metric of ST can be put in the Lorentz form η_{ab} by an appropriate choice of coordinates.
- *Weak Equivalence Principle*: Freely falling particles move on timelike geodesics of the ST. Equivalently: in a uniform gravitational field all objects, regardless of their composition, fall with precisely the same acceleration, which can also be stated as in a gravitational field the acceleration of a test particle is independent of its properties, including its rest mass.
- *Einstein's Equivalence Principle*: The outcome of any local, non-gravitational test experiment is independent of the experimental apparatus' velocity relative to the gravitational field and is independent of where and when in the gravitational field the experiment is performed.
- *Strong Equivalence Principle*: this is a version of the equivalence principle that applies to objects that exert a gravitational force on themselves, such as stars, planets, or black holes. It requires that the gravitational constant be the same everywhere in the universe and is incompatible with a fifth force. It is much more restrictive than the Einstein equivalence principle.

The Einstein's equivalent Principle is equivalent to the statement that if we could describe a physical interaction as a tensor relation in Special Relativity then this description should hold true in a locally inertial frame in a curved spacetime. This is a rule that can be simplistically stated as $, \rightarrow ;$ or standard derivatives extend into covariant ones. i.e. the law of conservation of particles in SR:

$$(nU^\alpha)_{,\alpha} = 0, \tag{28}$$

is converted in

$$(nU^\alpha)_{;\alpha} = 0, \tag{29}$$

Notice that the Einstein's principle is specific: (28) gets converted into (29).
And not into

$$(nU^{\alpha})_{;\alpha} = kR^2, \tag{30}$$

where R is the curvature scalar. Why? (3) would have physical implications: the curvature of spacetime would create particles. There is no evidence of such creation. Einstein's principle is backed by evidence. The law of conservation of entropy in Special Relativity is

$$U^{\alpha}S_{,\alpha} = 0, \tag{31}$$

But S is a scalar so this law would not change in a curved spacetime. And of course the law of conservation

$$T^{\mu\nu}{}_{,\nu} = 0, \tag{32}$$

transforms into

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{33}$$

with

$$T^{\mu\nu} = (\rho + p)U^{\mu}U^{\nu} + pg^{\mu\nu}, \tag{34}$$

where locally $g^{\mu\nu} \to \eta^{\mu\nu}$ in a local inertial frame.

Let's assume that we can represent the metric of our space time by

$$ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2). \tag{35}$$

We expect that far from the source $\phi = -GM/r$. Also we assume in all this that $|m\phi| \ll m$. We can compute the motion of a freely falling particle. If $\vec{p} = m\vec{U}$, where $\vec{U} = d\vec{x}/d\tau$ we will have:

$$\nabla_{\vec{U}}\vec{U} = 0 \tag{36}$$

If the proper time τ is the affine parameter along the geodesic so it is τ/m and we have:

$$\nabla_{\vec{p}}\vec{p} = 0 \tag{37}$$

which is also good for photons.

Let's look at the zero component:

$$m\frac{d}{d\tau}p^0 + \Gamma^0{}_{\alpha\beta}p^{\alpha}p^{\beta} = 0 \tag{38}$$

The particle is moving with $v \ll c$ so we can neglect the terms of the 4-velocity other p^0:

$$m\frac{d}{d\tau}p^0 + \Gamma^0{}_{00}(p^0)^2 = 0 \tag{39}$$

and

$$\Gamma^0{}_{00} = \phi_{,0} + O(\phi^2) \tag{40}$$

we get:

$$\frac{d}{d\tau}p^0 = -m\frac{\partial\phi}{\partial\tau} \tag{41}$$

The space components give:

$$p^\alpha p^i{}_{,\alpha} + \Gamma^i{}_{\alpha\beta}p^\alpha p^\beta = 0 \tag{42}$$

$$m\frac{dp^i}{d\tau} + \Gamma^i{}_{00}(p^0)^2 = 0 \tag{43}$$

which gives:

$$\frac{dp^i}{d\tau} = -m\phi_{,j}\delta^{ij}. \tag{44}$$

The geodesic equation can be written for $\vec{p}$

$$p^\alpha p_{\beta;\alpha} = 0 \tag{45}$$

this would give:

$$m\frac{dp_\beta}{d\tau} = \Gamma^\gamma{}_{\beta\alpha}p^\alpha p_\gamma \tag{46}$$

Which can also be written in terms of the metric:

$$m\frac{dp_\beta}{d\tau} = \frac{1}{2}g_{\nu\alpha,\beta}p_\nu p^\alpha \tag{47}$$

But if all the components of the metric are independent of x^β for all β, then p_β is a constant along the particle's trajectory.
Notice that if the metric does not depend on the time, we can find a coordinate system in which the metric components are time independent and p^0 will be conserved (this is the energy).
If we apply metric (36) to the definition of momentum we get:

$$\vec{p}\cdot\vec{p} = -m^2 = g_{\alpha\beta}p^\alpha p^\beta \tag{48}$$

$$= -(1+2\phi)(p^0)^2 + (1-2\phi)[(p^x)^2 + (p^y)^2 + (p^z)^2], \tag{49}$$

We can solve for p^0:

$$(p^0)^2 = [m^2 + (1-2\phi)(p^2)](1+2\phi)^{-1}, \tag{50}$$

where p^2 refers to the space coordinates. And we still assume $|\phi| \ll 1, |\mathbf{p}| \ll m$, so:

$$(p^0)^2 = m^2(1-2\phi) + (1-2\phi)(1-2\phi)p^2 \tag{51}$$
$$\approx m^2 - 2\phi m^2 + p^2 \tag{52}$$
$$p^0 \approx m(1 - 2\phi + \frac{p^2}{m^2})^{1/2} \tag{53}$$
$$\approx m(1 - \phi + \frac{p^2}{m^2}) \tag{54}$$

We can lower the index:

$$p_0 = g_{0\alpha}p^\alpha = -(1+2\phi)p^0 \tag{55}$$
$$= -(1+2\phi)m(1 - \phi + \frac{p^2}{m^2}) \tag{56}$$
$$-p_0 = m + m\phi + p^2/2m \tag{57}$$

The terms are the rest mass, the potential energy and the kinetic energy.

A general gravitational field will not be stationary in *any* frame, which means that we can not define a globally conserved energy.

We can now try looking at the result of the metric being axially symmetric (let's say it does not depend of an angle ψ):

$$p_\psi = g_{\psi\psi}p^\psi \approx g_{\psi\psi}md\psi/dt \approx mg_{\psi\psi}\Omega, \tag{58}$$

where Ω is the angular velocity of the particle. For a nearly flat metric,

$$g_{\psi\psi} = \vec{e}_\psi \cdot \vec{e}_\psi \approx r^2 \tag{59}$$

in cylindrical coordinates r, ψ, z so the conserved quantity is:

$$p_\psi \approx mr^2\Omega \tag{60}$$

8 Isometries and Killing vectors

Symmetries of the metric are called isometries.
For example

$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$ has several isometries.

These include translations:

$$x^\mu \to x^\mu + a^\mu \tag{61}$$

and Lorentz transformations:

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu \tag{62}$$

where $\Lambda^\mu{}_\nu$ is a Lorentz-transformation matrix. These are a total of ten isometries.

A systematic way of obtaining the isometries associated with a given metric are calculating the corresponding *Killing* vectors of it. Let's study the method.

A metric is invariant under the transformation $x^\alpha \to x'^\alpha$ if

$$g'_{ab}(x^\alpha) = g_{ab}(x^\alpha) \qquad for\ all\ coordinates\ x^\alpha \tag{63}$$

A transformation resulting in the above equation is called an isometry of the metric. Let's study how g_{ab} transforms.

$$g_{ab}(x) = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} g'_{cd}(x'), \tag{64}$$

Then using (63) if the transformation is an isometry we have

$$g_{ab}(x) = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} g_{cd}(x'), \tag{65}$$

Let's consider the simple situation in which the coordinates change is infinitesimal. Then, and also assuming that $x'^\alpha = x'^\alpha(x)$

$$x^a \to x'^a = x^a + \epsilon X^a(x) \tag{66}$$

where ϵ is small and arbitrary and X^a is a vector field. Differentiating we have

$$\frac{\partial x'^a}{\partial x^b} = \delta^a_b + \epsilon \frac{\partial X^a(x)}{\partial x^b} \tag{67}$$

Using (67) in (65) and expanding in power series up to first order we get

$$\begin{aligned} g_{ab}(x) =& (\delta^c_a + \epsilon\partial_a X^c)(\delta^d_b + \epsilon\partial_b X^d) g_{cd}(x^e + \epsilon X^e) \\ =& (\delta^c_a + \epsilon\partial_a X^c)(\delta^d_b + \epsilon\partial_b X^d)[g_{cd}(x) + \epsilon X^e \partial_e g_{cd} + ...] \\ =& g_{ab}(x) + \epsilon[g_{ad}\partial_b X^d + g_{bd}\partial_a X^d + X^e\partial_e g_{ab}] + O(\epsilon^2). \end{aligned} \tag{68}$$

which shows that to first order in ϵ

$$g_{ab}(x) - g_{ab}(x) = 0 = g_{ad}\partial_b X^d + g_{bd}\partial_a X^d + X^e\partial_e g_{ab}. \tag{69}$$

The right hand side is called the Lie derivative of the metric respect to the vector field $\vec{x}$ and it is

$$\mathcal{L}_{\vec{X}} g_{\mu\nu} = X^e\partial_e g_{ab} + g_{ad}\partial_b X^d + g_{bd}\partial_a X^d. \tag{70}$$

or

$$\mathcal{L}_{\vec{X}} g_{\mu\nu} = \nabla_e g_{ab} + \nabla_b X_a + \nabla_a X_b. \tag{71}$$

where ∇_μ is the covariant derivative calculated using $g_{\mu\nu}$. But the covariant derivative of the metric is by definition 0. So our equation defining isometries of the metric respect to a vector field $\vec{X}$ is or

$$\mathcal{L}_{\vec{X}} g_{\mu\nu} = \nabla_b X_a + \nabla_a X_b = 0. \tag{72}$$

If

$$\mathcal{L}_{\vec{X}} g_{\mu\nu} = 0, \tag{73}$$

then $\vec{X}$ is a Killing vector of the metric and it defines an isometry of the metric. i.e. for the Minkowski metric in standard Cartesian coordinates $\partial/\partial_{x^\alpha}$ where $\alpha = t, x, y, z$ are Killing vectors.

9 Gaussian coordinates

In an arbitrary spacetime manifold (not necessarily homogeneous or isotropic) we can do the following:

1. pick an initial spacelike hypersurface S_I,
2. place an arbitrary coordinate grid (x^1, x^2, x^3) on it,
3. look at the geodesic world lines orthogonal to it and attach to them:
4. coordinates $(x^1, x^2, x^3) = constant$, $x^0 \equiv t = t_I + \tau$ where τ is the proper time along the world line, with $\tau_{S_I} = 0$.

Now if we have a general $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ since $x^i = constant$ along the geodesics then $ds^2 = g_{00} dt^2$ along the geodesics.

But along the geodesics $ds^2 = -d\tau^2$ so $g_{00} = -1$ everywhere.

Let now $\vec{e}_\alpha$ be the coordinate basis vectors, and let $\vec{u} = d/d\tau$ be the tangent vector field to the geodesics (i.e. $\vec{u} = \vec{e}_0$). But by construction at $\tau = 0$:

$$\vec{u} \cdot \vec{e}_i = \vec{e}_0 \cdot \vec{e}_i = g_{0i} = 0 \tag{74}$$

and

$$\frac{d(\vec{u} \cdot \vec{e}_i)}{d\tau} = \nabla_{\vec{u}} (\vec{u} \cdot \vec{e}_i) = 0 + \vec{u} \cdot \nabla_{\vec{e}_i} \vec{u} \tag{75}$$

(the curves are geodesics so $\nabla_{\vec{u}} \vec{u} = 0$ and $\vec{e}_i$ and $\vec{u}$ form a coordinate basis $([\vec{e}_i, \vec{u}] = 0$. and because:

$$\vec{u} \cdot \nabla_{\vec{e}_i} \vec{u} = \frac{1}{2} \nabla_{\vec{e}_i} (\vec{u} \cdot \vec{u}) = 0 \tag{76}$$

and consequently $\vec{u} \cdot \vec{e}_i = g_{0i} = 0$ everywhere and we can write the metric in the so called synchronous form:

$$ds^2 = -dt^2 + g_{ij} dx^i dx^j \tag{77}$$