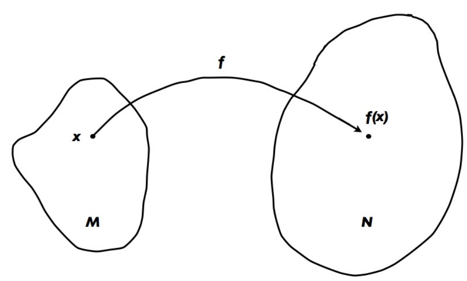
Introduction to General Relativity 2025
Lesson 5: Curved manifolds and Einstein's equations

Mario C Díaz

1 Mappings

A map f from a space M to a space N is a rule which associates with an element x of M a unique element y of N. The simplest example of a map is an ordinary real-valued function of $\Re$ into $\Re$.



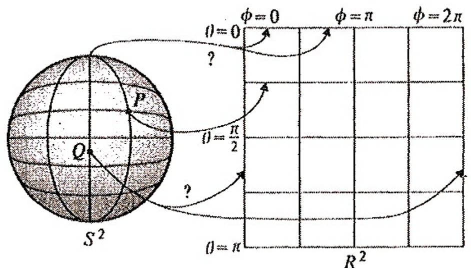
Note: the map gives a unique $f(x)$ for every x but there may not necessarily be a unique x for every $f(x)$.

2 Manifolds

- $\Re^n$ is the set of n-tuples of real numbers $(x_1, x_2, x_3, ..., x_n)$.
- A **manifold** is a set of points M which have an open neighborhood which has a continuous $1-1$ map onto an open set of $\Re^n$ for some n.
- We do not require a metric or anything else: i.e. there is no a priori geometrical notion associated with it. We do want the local topology of our space to be like $\Re^n$.
- In our definition we associated with a point P on M an n-tuple $(x_1(P), x_2(P), x_3(P), ..., x_n(P))$. These numbers $x_1(P), x_2(P), x_3(P), ..., x_n(P)$ are called the coordinates of P under the map.

3 Examples

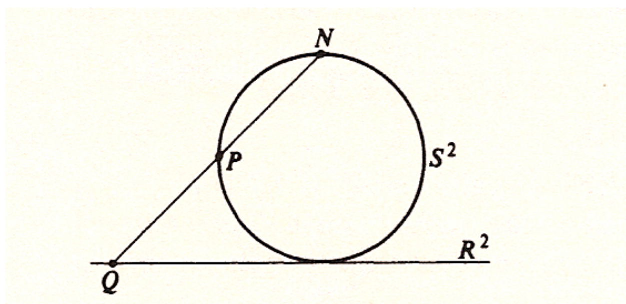
S^2, the sphere is a manifold.



from Schutz GMofMP

A map from S^2 to R^2, is good for points like P, but not for Q which it is mapped to $\theta = \pi/2, \phi = 0$ and $\theta = \pi/2, \phi = 2\pi$. And what about the pole?

The following is another map of S^2 called the stereographic map onto R^2. The map only fails at N.



from Schutz GMofMP

4 Other examples

1. The set of rotations of a rigid object in 3 dimensions is a manifold. (the three Euler angles provide the coordinates in R^3).

2. The set of all pure-boosts Lorentz transformations (the parameters are the three components of the velocity-boost).

3. The phase space defined by the position ($3N$ numbers) and the velocities ($3N$ numbers) of N particles is a manifold of dimension $6N$.

4. a vector space.

5. the set of all (x, y) solutions to a differential (or algebraic) equation for a function $y(x)$. A particular solution will be a curve on such manifold.

6. The first one is also a Lie group which is also a C^∞ manifold, i.e. $SO(3)$.

5 Differential Structure

- We will consider only ”differentiable manifolds”. This means that our manifolds will not only be continuous but also will have maps that can be differentiated with derivatives which will also be well defined and differentiable except for a few points.

- For example:
  The sphere is continuous and differentiable.
  A cone will not be differentiable at its vertex.

- Differentiability means we will be able to have vectors and one-forms and with then build all tensor types.
  Once we define a metric on the manifold we will have a correspondence between forms and vectors.

6 Review

1. A tensor field defines a tensor at every point.
2. Vectors and one-forms are linear operators on each other, producing real numbers.
3. Tensors are also linear operators on one-forms and vectors, producing real numbers.

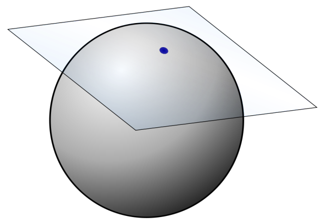
Tensor operations

1. multiplication by a scalar produces another tensor of the same kind.
2. addition of tensor components of the same type $\rightarrow$ produces another tensor of the same type.
3. multiplication of tensor of different types gives a new tensor of the sum of the types, the outer product of two tensors.
4. covariant differentiation of a tensor of type $\begin{pmatrix} m \\ n \end{pmatrix}$ gives a tensor of type $\begin{pmatrix} m \\ n+1 \end{pmatrix}$.
5. Contraction on a pair of indices of the components of a tensor of type $\begin{pmatrix} m \\ n \end{pmatrix}$ gives a tensor of type $\begin{pmatrix} m-1 \\ n-1 \end{pmatrix}$ (only between and upper and lower index).
6. any tensorial equation valid in one basis, is true in any other one.

7 Riemannian manifolds

Tangent space

In differential geometry, a tangent space can be defined at every point x of a differentiable manifold. This is a real vector space which we can pictured instinctively as containing the possible tangential directions that could be defined at a point x.
The elements of the tangent space are called tangent vectors at x. All the tangent spaces have the same dimension, equal to the dimension of the manifold.



The tangent space to a sphere (from Wikipedia commons)

A Riemannian manifold or Riemannian space (M, g) is a real differentiable manifold M in which each tangent space is equipped with an inner product given by g, a Riemannian metric, in a manner which varies smoothly from point to point. The metric g is a positive definite symmetric tensor: a metric tensor. This will let us define angles, lengths of curves, areas (or volumes), curvature, gradients of functions and divergence of vector fields. In a bit more rigorous but not tremendously obscure jargon a Riemannian manifold is a differentiable manifold in which the tangent space at each point is a finite-dimensional Euclidean space. The word is honoring the German mathematician Bernhard Riemann. In reality we will be using pseudo-Riemannian manifolds because the metric will not be positive definite. We will be using a metric with signature $(- + ++)$ like the Minkowski metric. These metrics will let us have in the tangent space at any point vectors which could have positive, negative or zero magnitude. But our metric will still be symmetric.

8 The metric and the tangent space

When Schutz talks about "local flatness" he is describing the same notion that we used in our definition of manifold: i.e that any point of the manifold will have a tangent space associated at the point that we associate naturally to it. The idea is that with a metric we have a unique way to associate a one-form with a vector and vice versa. Why do we say that the metric signature has to be $(- + ++)$? The reason is that we want our tangent space not to be "Euclidean" but "Minkowskian". This means that we want to have a symmetric metric which can by linear algebra theorems be diagonalized to have eigenvalues (-1,1,1,1). The sum of the diagonal elements is called the signature. So we want our metric to have signature $+2$.

Another important point is that in order to have coordinate basis we need matrices $\Lambda\alpha'_{\beta}$ that can be turned into a Minkowski metric. This means that we need transformations:

$$\frac{\partial \Lambda\alpha'_{\beta}}{\partial x^{\gamma}} = \frac{\partial \Lambda\alpha'_{\gamma}}{\partial x^{\beta}}$$

9 The local flatness theorem

We will state as Schutz' book does, the following theorem which I will not prove in class. But I strongly recommend reading it.

At any point $\mathcal{P}$ in a pseudo riemannian manifold we can find a coordinate system$\{X^\alpha\}$ with origin at $\mathcal{P}$ where

$$g_{\alpha\beta}(X^\mu) = \eta_{\alpha\beta} + O[(x^\mu)^2] \tag{1}$$

In an equivalent way:

$$g_{\alpha\beta}(\mathcal{P}) = \eta_{\alpha\beta} \;\; for \;\; all \;\; \alpha, \;\; \beta, \tag{2}$$

$$\frac{\partial}{\partial x^\gamma} g_{\alpha\beta}(\mathcal{P}) = 0 \;\; for \;\; all \;\; \alpha, \;\; \beta, \;\; \gamma; \tag{3}$$

but beware:

$$\frac{\partial^2}{\partial x^\gamma \partial x^\mu} g_{\alpha\beta}(\mathcal{P}) \neq 0 \tag{4}$$

10 Lengths and ...

We recall that the metric $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ gives us the line element ds. i.e. we get $dl \equiv |g_{\alpha\beta} dx^\alpha dx^\beta|^{1/2}$, with which we can calculate the length of a curve:

$$l = \int_{\substack{along \\ curve}} |g_{\alpha\beta} dx^\alpha dx^\beta|^{1/2} \tag{5}$$

$$l = \int_{\lambda_0}^{\lambda_1} \left| g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right|^{1/2} \tag{6}$$

where λ is the parameter of the curve. We can also define a curve by the vector field tangent to it. If we have $V^\alpha = dx^\alpha / d\lambda$ we get,

$$l = \int_{\lambda_0}^{\lambda_1} |\vec{V} \cdot \vec{V}|^{1/2} d\lambda \tag{7}$$

11 ... and Volumes

For the volume which is 4-D we have:

$$dx^0 dx^1 dx^2 dx^3 = \frac{\partial(x^0, x^1, x^2, x^3)}{\partial(x^{0'}, x^{1'}, x^{2'}, x^{3'})} dx^{0'} dx^{1'} dx^{2'} dx^{3'}, \tag{8}$$

Where the indicated quotient is the Jacobian of the transformation. But if we use the local flatness theorem, all we need to see is that t in the tangent space this Jacobian will be the product of the determinant of the transformation times the determinant of the Minkowski metric times the determinant of the transpose of the matrix of coordinates transformations. This result essentially says that the determinant is the determinant of the Minkowski metric which takes us to the following result, a **very important one**:

$$dx^0 dx^1 dx^2 dx^3 = [-det(g_{\alpha'\beta'})]^{1/2} dx^{0'} dx^{1'} dx^{2'} dx^{3'}, \tag{9}$$

$$= (-g)^{1/2} dx^{0'} dx^{1'} dx^{2'} dx^{3'}. \tag{10}$$

This gives the proper volume element.

Simple example

You can compare with the metric in spherical coordinates, $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ If we calculate $dV = dx^3$ in cartesian it would give, remembering that the determinant is $r^4 \sin^2\theta$ (watch out: the metric here is 3-D positive definite):

$$dV = r^2 \sin\theta dr d\theta d\phi \tag{11}$$

12 Covariant Differentiation again

Differentiation of a vector in more than one dimension implies to look at the value of a vector at given point and then after choosing a given direction (i.e. the direction along which we want to calculate the derivative of the vector) at the value it has at a different point (in that given direction), located from the previous one a distance that we make infinitesimal.
This infinitesimal direction would be a fuzzy concept when the space is not flat.

That's the reason why we need the concept of covariant derivation.
Of course in a very small neighborhood of a given point when we move around in the tangent space, the covariant derivative will be the same as the ordinary derivative.

13 The divergence of a vector

$$V^{\alpha}{}_{;\alpha} = V^{\alpha}{}_{,\alpha} + \Gamma^{\alpha}_{\mu\alpha} V^{\mu} \tag{12}$$

And,

$$\Gamma^{\alpha}{}_{\mu\alpha} = \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu,\alpha} + g_{\beta\alpha,\mu} - g_{\mu\alpha,\beta}) \tag{13}$$

$$= \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu,\alpha} - g_{\mu\alpha,\beta}) + \frac{1}{2} g^{\alpha\beta} g_{\beta\alpha,\mu} \tag{14}$$

Now notice:

$$g^{\alpha\beta} g_{\beta\mu,\alpha} - g^{\alpha\beta} g_{\mu\alpha,\beta} = g_{\alpha\mu,\alpha} - g_{\mu\beta,\beta} = 0 \tag{15}$$

(the latter one due to the symmetry of the metric tensor!) which means:

$$\Gamma^{\alpha}{}_{\mu\alpha} = \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta,\mu} \tag{16}$$

Let's review some properties of matrices: The inverse of

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is } M^{-1} = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where $(ad - bc)$ is the matrix determinant. The x derivative of M is $M,x = \begin{pmatrix} a,x & b,x \\ c,x & d,x \end{pmatrix}$. Calculating the trace of $M^{-1}M,x$:

$$Tr\left(M^{-1}M,x\right) = Tr\left(\frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a,x & b,x \\ c,x & d,x \end{pmatrix}\right)$$

$$= Tr\left(\frac{1}{(ad-bc)}\left(\begin{array}{cc} da_{,x}-bc_{,x} & db_{,x}-bd_{,x} \\ -ca_{,x}+ac_{,x} & -cb_{,x}+ad_{,x} \end{array}\right)\right)$$

i.e., $Tr\left(M^{-1}M,x\right) = \frac{(da_{,x}+ad_{,x}-bc_{,x}+-cb_{,x})}{(ad-bc)} = \frac{(ad-bc)_{,x}}{(ad-bc)}$ so

$$Tr\left(M^{-1}M,x\right) = (\log(det(M))_{,x}$$

This is a general result for a nonsingular square matrix. So we can apply it to our metric to obtain:

$$[log(det||g_{\alpha\beta}||)]_{,\alpha} = Tr(||g_{\alpha\beta}||^{-1}||g_{\mu\nu,\alpha}||)$$

which means:

$$\frac{g_{,\alpha}}{g} = g^{\mu\nu}g_{\mu\nu,\alpha} \text{ and then } g_{,\alpha} = gg^{\mu\nu}g_{\mu\nu,\alpha} \tag{17}$$

Using this result in (16):

$$\Gamma^{\alpha}{}_{\mu\alpha} = (\log(-g)^{1/2}))_{,\mu} = \frac{(\sqrt{-g})_{,\mu}}{\sqrt{-g}} \tag{18}$$

And then the divergence can be written:

$$V^{\alpha}{}_{;\alpha} = V^{\alpha}{}_{,\alpha} + V^{\alpha}\frac{(\sqrt{-g})_{,\alpha}}{\sqrt{-g}} \tag{19}$$

14 Gauss' law

We can take now a look at Gauss' law where we integrate the divergence over a volume (using the proper volume element).

$$\int V^{\alpha}{}_{;\alpha}\sqrt{-g}d^4x = \int(\sqrt{-g}V^{\alpha})_{,\alpha}d^4x. \tag{20}$$

And then,

$$\int(\sqrt{-g}V^{\alpha})_{,\alpha}d^4x = \oint V^{\alpha}n_{\alpha}\sqrt{-g}d^3S. \tag{21}$$

So Gauss' law on a curved manifold is:

$$\int V^{\alpha}{}_{;\alpha}\sqrt{-g}d^4x = \oint V^{\alpha}n_{\alpha}\sqrt{-g}d^3S. \tag{22}$$

where $\vec{n}$ is a 4-vector normal to the 3-surface element d^3S.

15 Curvature

We need to make a difference between extrinsic and intrinsic curvature.

We define **Extrinsic curvature** for a space embedded in another space and it is related to the radius of curvature of circles that "are" in the embedded space. For example the extrinsic curvature of a circle is equal to the inverse of its radius everywhere.

Intrinsic curvature, instead, is defined at each point in a Riemannian manifold.
An intrinsic definition of the Gaussian curvature at a point P could be the following: imagine the ant that Schutz describes in his book, which is tied to P with a short thread of length r. She runs around P while the thread is completely stretched and measures the length C(r) of one complete trip around P.

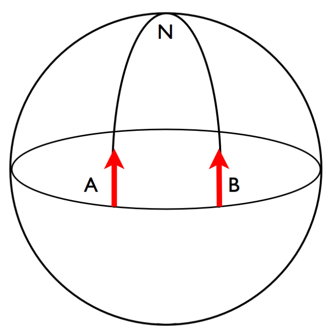
If the surface were flat, she would find $C(r) = 2\pi r$.

On curved surfaces, the formula for C(r) will be different, and the Gaussian curvature K at the point P can be computed by the so-called Bertrand - Diquet - Puiseux theorem as:

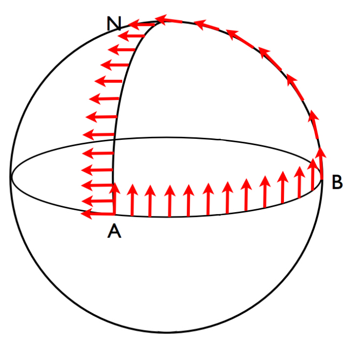
$$K = \lim_{r \to 0} (2\pi r - C(r)) . \frac{3}{\pi r^3} . \qquad (23)$$

When we apply this formula to a a cylinder we get 0.

A clear example of intrinsic non-zero curvature is the failure of the Euclid's' fifth postulate. i.e we take a look at two vectors that are parallel to each other at the equator: the curves to which they are tangent will not have its tangent vectors parallel at the north pole.



And if we try to transport one the vectors parallel to itself through a closed loop like in the figure, i.e. from **A** to **B** and then along a meridian the same way (parallel) through it until it reaches the north pole , and then keeping it parallel to itself again down another meridian that passes through the point **A** at the equator it ends up at $90°$ with itself at the origin.



16 Parallel transport and geodesics

The construction we made in the previous slide is called parallel transport.
When the covariant derivative of a vector field respect to itself is zero, the curves to which this vector field is tangent are called **geodesics**. It is also said that a vector field to geodesic curves is parallel transported to itself along these geodesics. For these lines the vector tangent to the curve at one point is parallel to the vector tangent to the same curve at a previous point.

The equation that describes such curves can be obtained by writing the equation:

$$\nabla_{\vec{U}}\vec{U} = 0$$

This can be written:

$$U^{\beta}U^{\alpha}{}_{;\beta} = U^{\beta}U^{\alpha}{}_{,\beta} + \Gamma^{\alpha}{}_{\mu\beta}U^{\mu}U^{\beta} = 0 \tag{24}$$

If we let s be a parameter along the curve, then $U^{\alpha} = dx^{\alpha}/ds$ and $U^{\beta}\partial/\partial x^{\beta} = d/ds$ from where we get:

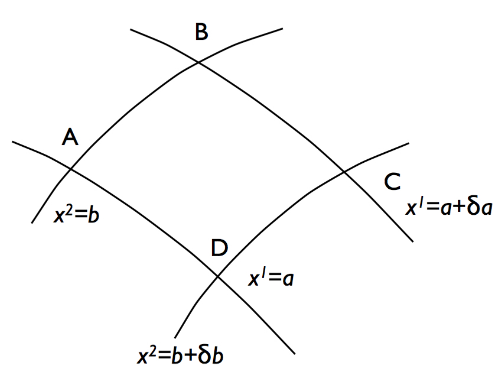
$$\frac{d}{ds}\left(\frac{dx^{\alpha}}{ds}\right) + \Gamma^{\alpha}{}_{\mu\beta}\frac{dx^{\mu}}{ds}\frac{dx^{\beta}}{ds} = 0 \tag{25}$$

This is called the **geodesic equation**.
If s is a parameter along the geodesic curve, it can be re-parametrized the following way: $\lambda = as + b$.

Parameters related this way are called affine. A geodesic is a curve of extremal length between two points. This is the generalization of the fact that the extremal (minimal) length between two point in Euclidean space is given by a straight line. It can be shown that the proper distance along a geodesic is an affine parameter itself and that is the way geodesics are typically calculated.

17 Intrinsic curvature and the curvature tensor

We want to parallel transport a vector $\vec{V}$ defined at A all through the closed loop A-B-C-D in the figure below,



and then calculate the difference experienced throughout the transport process.

The result of parallel transporting, by definition, has to be $\nabla_{\vec{e}_1}\vec{V} = 0$ which gives:

$$\frac{\partial V^\alpha}{\partial x^1} = -\Gamma^\alpha{}_{\mu 1} V^\mu \tag{26}$$

where $\vec{e}_1$ is the particular direction taken to transport the vector. The difference after a full loop should be:

$$\delta V = \delta V^\alpha(B-A) + \delta V^\alpha(C-B) + \delta V^\alpha(D-C) + \delta V^\alpha(D-A) \tag{27}$$

Starting first with $\delta V^\alpha(B-A)$ we can calculate it by summing (integrating) its change along the first step of the loop:

$$\delta V^\alpha(B-A) = V^\alpha(B) - V^\alpha(A) = \int_A^B \frac{\partial V^\alpha}{\partial x^1} dx^1 \tag{28}$$

$$= -\int_{x^2=b} \Gamma^\alpha{}_{\mu 1} V^\mu dx^1 \tag{29}$$

Similarly for the rest of the loop we get:

$$\delta V^\alpha(C-B) = -\int_{x^1=a+\delta a} \Gamma^\alpha{}_{\mu 2} V^\mu dx^2$$

$$\delta V^\alpha(D-C) = \int_{x^2=b+\delta b} \Gamma^\alpha{}_{\mu 1} V^\mu dx^1$$

and back to A again,

$$\delta V^\alpha(D-A) = \int_{x^1=a} \Gamma^\alpha{}_{\mu 2} V^\mu dx^2 \tag{30}$$

The difference in signs are due to going in the x^1 and x^2 first and then along $-x^1$ and $-x^2$ when closing the loop. Adding all the equations we have the final vector:

$$\delta V^\alpha = \int_{x^1=a} \Gamma^\alpha{}_{\mu 2} V^\mu dx^2 + \int_{x^2=b+\delta b} \Gamma^\alpha{}_{\mu 1} V^\mu dx^1 \tag{31}$$

$$- \int_{x^1=a+\delta a} \Gamma^\alpha{}_{\mu 2} V^\mu dx^2 - \int_{x^2=b} \Gamma^\alpha{}_{\mu 1} V^\mu dx^1 \tag{32}$$

Notice that these terms do not cancel because $\Gamma^\alpha{}_{\mu\nu}$ and V^μ are not constant along the loop (we are assuming curvature). We can now group the integrals by the paths they follow. Along the x^1 one direction we have

$$\delta V^\alpha(x^1) = \int_{x^1=a} \Gamma^\alpha{}_{\mu 2} V^\mu dx^2 - \int_{x^1=a+\delta a} \Gamma^\alpha{}_{\mu 2} V^\mu dx^2$$

$$= \int_{x^1} \Gamma^\alpha{}_{\mu 2} V^\mu - \Gamma^\alpha{}_{\mu 2} V^\mu dx^2 \tag{33}$$

We can now multiply and divide the integrand by $\left(x^1(B) - x^1(A)\right) = \delta a$ and take the limit when $\delta a \to \infty$ using

$$\lim_{(x^1(B)-x^1(A))\to\infty} \left\{ \frac{\Gamma^\alpha{}_{\mu 2} V^\mu(x^1(B)) - \Gamma^\alpha{}_{\mu 2} V^\mu(x^1(A))}{x^1(B) - x^1(A)} \right\} = \frac{\partial}{\partial x^1}\left(\Gamma^\alpha_{\mu 1} V^\mu\right) \tag{34}$$

and then we get for the stretch back and forth in the x^1 direction.

$$\begin{aligned}\delta V^\alpha(x^1) &= \int_{x^1} \frac{\partial}{\partial x^1}\left(\Gamma^\alpha_{\mu 1} V^\mu\right)\left(x^1(B) - x^1(A)\right) dx^2 \\ &= \int_{x^1} \frac{\partial}{\partial x^1}\left(\Gamma^\alpha_{\mu 1} V^\mu\right)\delta a dx^2 \end{aligned} \tag{35}$$

For the entire loop we get then,

$$\delta V^\alpha \simeq -\int_b^{b+\delta b} \delta a \frac{\partial}{\partial x^1} \Gamma^\alpha{}_{\mu 2} V^\mu dx^2 + \int_a^{a+\delta a} \delta b \frac{\partial}{\partial x^2} \Gamma^\alpha{}_{\mu 1} V^\mu dx^1$$

Where we use $\left(x^2(D) - x^2(C)\right) = \delta b$

Now to perform the integration we take advantage of the fact that while the vector is changing in one direction we perform the integral in the other one. But we can see that the integrands do not depend in one case on x^2 and in the other independent direction on x^1 which are the integration variables as we move along the entire loop:

which gives:

$$\delta V^\alpha \approx -\delta b \delta a \frac{\partial}{\partial x^1} \Gamma^\alpha{}_{\mu 2} V^\mu + \delta a \delta b \frac{\partial}{\partial x^2} \Gamma^\alpha{}_{\mu 1} V^\mu = \delta a \delta b \left[-\frac{\partial}{\partial x^1} \Gamma^\alpha{}_{\mu 2} V^\mu + \frac{\partial}{\partial x^2} \Gamma^\alpha{}_{\mu 1} V^\mu \right] \tag{36}$$

We perform the derivatives of the products and proceed to eliminate the derivatives of V^α using (26) getting:

$$\delta V^\alpha = \delta a \delta b \left[\Gamma^\alpha{}_{\mu 1,2} - \Gamma^\alpha{}_{\mu 2,1} + \Gamma^\alpha{}_{\nu 2} \Gamma^\nu{}_{\mu 1} - \Gamma^\alpha{}_{\nu 1} \Gamma^\nu{}_{\mu 2}\right] V^\mu. \tag{37}$$

Here 1 and 2 are antisymmetric because the indices refer to opposite paths. If we used general coordinate lines x^σ and x^λ, $\delta V^\alpha = \{change\ \ of\ \ V^\alpha\}$ is:

$$1)\ \ \delta a \rightarrow \vec{e}_\sigma, 2)\ \ \delta b \rightarrow \vec{e}_\lambda, 3) - \delta a \rightarrow \vec{e}_\sigma, and\ \ finally\ \ 4) - \delta b \rightarrow \vec{e}_\lambda$$

or in better mathematical language:

$$\delta V^\alpha = \delta a \delta b \left[\Gamma^\alpha{}_{\mu\sigma,\lambda} - \Gamma^\alpha{}_{\mu\lambda,\sigma} + \Gamma^\alpha{}_{\nu\lambda} \Gamma^\nu{}_{\mu\sigma} - \Gamma^\alpha{}_{\nu\sigma} \Gamma^\nu{}_{\mu\lambda}\right] V^\mu. \tag{38}$$

This means that δV^α is proportional to V^μ the proportionality factors being $\delta a \delta b$ and the number –after all is a function evaluated at the loop, and consequently a number– $[\Gamma^\alpha{}_{\mu\sigma,\lambda} - \Gamma^\alpha{}_{\mu\lambda,\sigma} + \Gamma^\alpha{}_{\nu\lambda} \Gamma^\nu{}_{\mu\sigma} - \Gamma^\alpha{}_{\nu\sigma} \Gamma^\nu{}_{\mu\lambda}]$. We call the function in general the Riemann curvature tensor $R^\alpha_{\beta\mu\nu}$.

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\sigma\mu} \Gamma^\sigma{}_{\beta\nu} - \Gamma^\alpha{}_{\sigma\nu} \Gamma^\sigma{}_{\beta\mu}. \tag{39}$$

The Riemann tensor "measures" how much a vector transported parallel to itself around a close loop in a given manifold, has departed from being parallel. From its definition we see that if the manifold is flat the vector remains parallel to itself after this transport.

We can look at another way of deriving it.
We will use the fact that in general covariant derivation is not commutative.
We can define the commutator of covariant derivatives this way:

$$\nabla_c \nabla_d T^{a\ldots}_{b\ldots} - \nabla_d \nabla_c T^{a\ldots}_{b\ldots}.$$

Let us calculate the covariant derivative of a vector X^a:

$$\nabla_c X^a = \partial_c X^a + \Gamma^a{}_{bc} X^b.$$

Then using that it is a tensor $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$:

$$\nabla_d \nabla_c X^a = \partial_d(\partial_c X^a + \Gamma^a{}_{bc} X^b) + \Gamma^a{}_{ed}(\partial_c X^e + \Gamma^e{}_{bc} X^b) - \Gamma^e{}_{cd}(\partial_e X^a + \Gamma^a{}_{be} X^b),$$

and,

$$\nabla_c \nabla_d X^a = \partial_c(\partial_d X^a + \Gamma^a{}_{bd} X^b) + \Gamma^a{}_{ec}(\partial_d X^e + \Gamma^e{}_{bd} X^b) - \Gamma^e{}_{dc}(\partial_e X^a + \Gamma^a{}_{be} X^b),$$

Now we can subtract the two equations and assuming that:

$$\partial_c \partial_d X^a = \partial_d \partial_c X^a$$

get:

$$\nabla_c \nabla_d X^a - \nabla_d \nabla_c X^a = R^a{}_{bcd} X^b + (\Gamma^e{}_{cd} - \Gamma^e{}_{dc}) \nabla_e X^a, \tag{40}$$

where we have defined $R^a{}_{bcd}$:

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^e{}_{bd} \Gamma^a{}_{ec} - \Gamma^e{}_{bc} \Gamma^a{}_{ed}. \tag{41}$$

Notice that when the metric is torsion free the parenthesis on the right hand side of (36) vanishes, so we get:

$$\nabla_{[c} \nabla_{d]} X^a = \frac{1}{2} R^a{}_{bcd} X^b. \tag{42}$$

Notice that the way we define it the Riemann tensor is a $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ type tensor. We can describe it by saying that it ”measures” the ”un-commutativity” of the covariant derivative. Now we want to calculate $\mathbf{R}$ in terms of the metric (after all the metric is the measuring element in our manifolds). We can do it in a locally inertial frame (a frame in which by definition the connection coefficients vanish and the covariant derivative is the regular derivative).

The connection coefficients at a given point $\mathcal{P}$, can be computed using the eq (13) from slide 23 in this Lesson.

$$\Gamma^a{}_{bc,d} = \frac{1}{2} g^{ae}(g_{eb,cd} + g_{ec,bd} - g_{bc,ed}), \tag{43}$$

and then we get at $\mathcal{P}$:

$$R^a{}_{bcd} = \frac{1}{2} g^{ae}(g_{eb,dc} + g_{ed,bc} - g_{bd,ec} - g_{eb,cd} + g_{ec,bd} + g_{bc,ed}).$$

But we're working with symmetric **g** so $g_{eb,dc} = g_{eb,cd}$,
and then at $\mathcal{P}$:

$$R^a{}_{bcd} = \frac{1}{2} g^{ae} (g_{ed,bc} - g_{ec,bd} + g_{bc,ed} - g_{bd,ec}). \tag{44}$$

We can lower the index a:

$$R_{abcd} = g_{ae} R^e{}_{bcd} = \frac{1}{2} (g_{ad,bc} - g_{ac,bd} + g_{bc,ad} - g_{bd,ac}). \tag{45}$$

And now it is easy to verify:

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab} \tag{46}$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0. \tag{47}$$

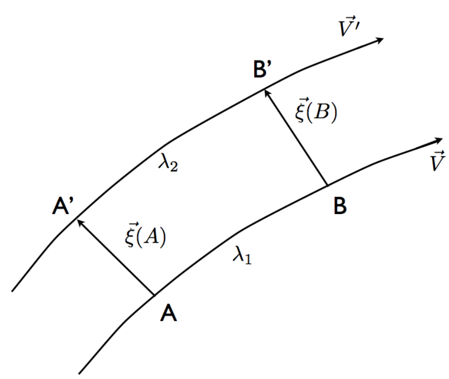
R_{abcd} is antisymmetric on the first and second pair of indices and symmetric on exchange of the two pairs. Notice that (40) and (41) are true tensor equations, while (38) is not (it does not include covariant derivatives: it is only valid in that particular frame).

$$R_{abcd} = 0 \Leftrightarrow flat\ manifold. \tag{48}$$

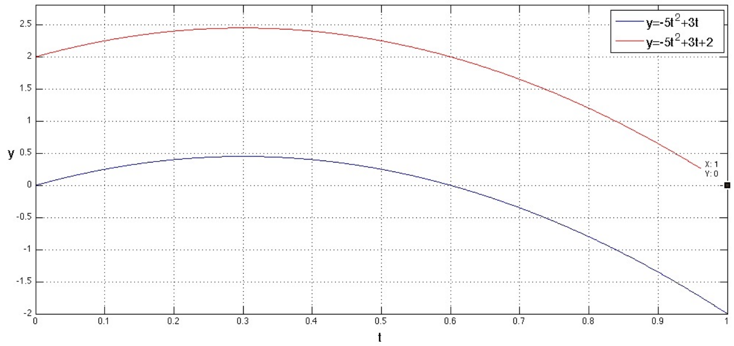
18 Geodesic deviation... Or how dead is Euclides' Fifth postulate

(The following narrative follows d'Inverno's book)

Let's look at two geodesic curves. They are both solutions (i.e. belong to the same family) of the geodesic equation. λ is the affine parameter. $\vec{V}$ and $\vec{V'}$ are the respective tangent vectors to each of them. We define now $\vec{\xi}$ which is a vector that connects both, i.e. from a given value of λ on one to the similar value of λ on the other one (λ, the affine parameter could well be the proper time).



This in the same sense that $(x = t, y = -5t^2 + 3t)$ and $(x = t, y = -5t^2 + 3t + 2)$ are solutions of the geodesic equation for a uniform potential.



We want to investigate how much do one depart form the other. Let's look at 2-surface S spanned by what is called a congruence of (timelike) geodesics. This is a family of geodesics such that exactly one of the curves goes through every point of S. The parametric equation of S is:

$$x^a = x^a(\tau, \lambda), \quad \text{So then:} \tag{49}$$

$$V^a = \frac{dx^a}{d\tau} \qquad \xi^a = \frac{dx^a}{d\lambda} \tag{50}$$

i.e. V^a is the tangent vector to the time-like geodesic at each point and ξ^a is a connecting vector connecting two neighboring curves in the congruence.

The commutator of V^a and ξ^a satisfies:

$$[V, \xi]^a = V^b \partial_b \xi^a - \xi^b \partial_b V^a \tag{51}$$

$$= \frac{dx^b}{d\tau} \frac{\partial}{\partial x^b} \left(\frac{dx^a}{d\lambda} \right) - \frac{dx^b}{d\lambda} \frac{\partial}{\partial x^b} \left(\frac{dx^a}{d\tau} \right) \tag{52}$$

$$= \frac{d}{d\tau} \left(\frac{dx^a}{d\lambda} \right) - \frac{d}{d\lambda} \left(\frac{dx^a}{d\tau} \right) \tag{53}$$

$$= \frac{d^2 x^a}{d\tau d\lambda} - \frac{d^2 x^a}{d\lambda d\tau} = 0 \tag{54}$$

But we can (we should) replace these derivatives by a covariant derivative:

$$\nabla_V \xi^a - \nabla_\xi V^a = 0 \tag{55}$$

19 Lie derivative

If we now take the covariant derivative of the previous equation respect to V^a (which will mean we want to see how this equation varies along the geodesics):

$$\nabla_V \nabla_V \xi^a - \nabla_V \nabla_\xi V^a = 0 \tag{56}$$

This suggests the following definition:
For any given two vector fields, we can always define the *Lie bracket* $[\vec{U},\vec{V}]$:

$$[\vec{U},\vec{V}]^{\alpha} = U^{\beta}\nabla_{\beta}V^{\alpha} - V^{\beta}\nabla_{\beta}U^{\alpha}. \tag{57}$$

Notice that:

$$[\vec{U},\vec{V}]^{\alpha} = U^{\beta}V^{\alpha}{}_{;\beta} - V^{\beta}U^{\alpha}{}_{;\beta} \tag{58}$$

$$[\vec{U},\vec{V}]^{\alpha} = U^{\beta}V^{\alpha}{}_{,\beta} + U^{\beta}\Gamma^{\alpha}{}_{\delta\beta}V^{\delta} - V^{\beta}U^{\alpha}{}_{,\beta} - V^{\beta}\Gamma^{\alpha}{}_{\delta\beta}U^{\delta} \tag{59}$$

$$= U^{\beta}V^{\alpha}{}_{,\beta} - V^{\beta}U^{\alpha}{}_{,\beta} \tag{60}$$

which shows that it does not depend on the metric.

This bracket is also called the *Lie derivative* of $\vec{Y}$ along $\vec{X}$.

$$\mathcal{L}_{\vec{U}}\vec{V} = [\vec{U},\vec{V}] \tag{61}$$

The Lie derivative can also be defined for a tensor in general:

$$\mathcal{L}_{\vec{X}}T^{a\ldots}_{b\ldots} = X^{c}\partial_{c}T^{a\ldots}_{b\ldots} - T^{c\ldots}_{b\ldots}\partial_{c}X^{a} + T^{a\ldots}_{c\ldots}\partial_{b}X^{c} + \cdots . \tag{62}$$

Notice that the Lie derivative is linear, is Leibniz, it is "type preserving", i.e. the Lie derivative of a tensor of type $\begin{pmatrix} p \\ q \end{pmatrix}$ is another tensor of type $\begin{pmatrix} p \\ q \end{pmatrix}$.

Back to the study of geodesic deviation. We will use the following result:

$$\nabla_{X}(\nabla_{Y}Z^{a}) - \nabla_{Y}(\nabla_{X}Z^{a}) - \nabla_{[X,Y]}Z^{a} = R^{a}{}_{bcd}Z^{b}X^{c}Y^{d} \tag{63}$$

If we set $X^{a} = Z^{a} = V^{a}$ and $Y^{a} = \xi^{a}$ then the third term vanishes because of our previous result regarding the Lie derivative of ξ respect to$V$, but $\nabla_{\vec{V}}\vec{V} = 0$ so the second term also vanishes.

So we get:

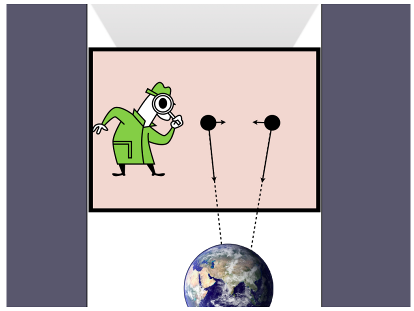
$$\nabla_{V}(\nabla_{\xi}V^{a}) = R^{a}{}_{bcd}V^{b}V^{c}\xi^{d} \tag{64}$$

We can now use (53) which will gives us, valid everywhere:

$$\nabla_{V}(\nabla_{V}\xi^{a}) = R^{a}{}_{bcd}V^{b}V^{c}\xi^{d} \tag{65}$$

This means that while geodesics in a flat space-time remain parallel (maintain their separation constant), geodesics in a curved manifold do not in general. The Riemman tensor is actually a "measure" of how much they don't.

In Lesson 6 we study the situation seen when dropping two particles inside a lift falling down freely through a shaft near the earth. The particles are getting closer together and this is how we know there is a gravitational field. This is a central idea leading to a theory of relativity that needs geometry in its formulation: The paths followed by test particles will indicate the curvature-geometry of space.If we call *geodesics* these paths, how these paths diverge or converge, i.e. how they deviate from parallelism is what indicates the presence of a gravitational field.



20 The Ricci and the Einstein tensor

If we look at equation (41) we can keep working in an inertial frame and taking one more derivative get:

$$R_{abcd,e} = \frac{1}{2}(g_{ad,bce} - g_{ac,bde} + g_{bc,ade} - g_{bd,ace}). \tag{66}$$

Using the symmetry of $g_a b$ and the commutativity of the partial derivatives:

$$R_{abcd,e} + R_{abec,d} + R_{abed,c} = 0 \tag{67}$$

But we can generalize:

$$R_{abcd;e} + R_{abec;d} + R_{abed;c} = 0 \tag{68}$$

which as a tensor equation is valid in any system and receives the name of: Bianchi identity.

There is only one possible contraction with the Riemann tensor that defines a very important derived tensor, the Ricci tensor:

$$R_{bc} = R^a{}_{bac} \tag{69}$$

Other contractions are possible as well,but due to the symmetries of the Riemann tensor they will vanish or give $-R_{bc}$.

We can also contract further and define the Ricci scalar:

$$R = g^{bc}R_{bc} = g^{cd}g^{ab}R_{acbd} \tag{70}$$

We can also explore a contraction similar to the one performed to get the Ricci tensor with the Bianchi identities.

$$g^{ac}[R_{abcd;e} + R_{abec;d} + R_{abed;c}] = 0 \tag{71}$$

To do this calculation we work each term of the sum this way:

$$g^{ac}R_{abcd;e} = (g^{ac}R_{abcd})_{;e} - g^{ac}{}_{;e}R_{abcd} \tag{72}$$

But $g^{ac}{}_{;e} = 0$ so we get: $[R_{bd;e} + R_{be;d} + R^c{}_{bed;c}] = 0$ Contracting again with g:

$$g^{bd}[R_{bd;e} + R_{be;d} + R^c{}_{bed;c}] = R_{;e} - R^c{}_{e;c} - R^c{}_{e;c} = 0 \tag{73}$$

Since R is a scalar

$$(2R^c{}_e - \delta^c{}_e R)_{;e} = 0 \tag{74}$$

We define now, the Einstein tensor:

$$G^{ab} \equiv R^{ab} - g^{ab}R = G^{ba} \tag{75}$$

Then the previous equation gives:

$$G^{ab}{}_{;b} = 0 \tag{76}$$

21 Summary

1. Physics will take place on Riemmanian manifolds , smooth spaces that locally resemble R^4 with a metric defined on them.

2. The metric has signature $+2$, i.e. $(- + ++)$ and there always exists a coordinate system in which, at a single point, we can have

$$g_{ab} = \eta_{ab}, \tag{77}$$

$$g_{ab,e} = 0 \Rightarrow \Gamma^a{}_{be} = 0 \tag{78}$$

3. The element of proper volume is:

$$|g|^{1/2} d^4x, \tag{79}$$

where g is the determinant of the metric.

4. The covariant derivative is the extension of the regular definition of derivative accounting for curvature.

5. When the covariant derivative of a tensor along a curve is zero we will interpret saying that the quantity has being transported parallel. A geodesic is a curve that transport its tangent vector parallel to itself. Its affine parameter can be taken to be the proper distance itself.

6. The Riemman tensor encodes all the information about the curvature of the manifold. It only vanishes identically when the manifold is flat. It has 20 independent components, and it satisfies the Bianchi identities, differential equations. It depends on the metric and its first and second partial derivatives. The Ricci scalar, the Ricci tensor, and the Einstein tensor are derived quantities constructed with contractions of the Riemman tensor. The Einstein tensor codifies the entire geometric content interacting with matter and energy to give the dynamics of the gravitational field, via the Einstein's equations:

$$G^{\mu\nu} = 8\pi T^{\mu\nu} \tag{80}$$

where $T^{\mu\nu}$ is the energy momentum tensor. As a result of this and (75) we have

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{81}$$

which means that the quadri-divergence of the energy momentum tensor is a conserved quantity.

22 The weak field limit

Far away enough from the source a gravitational field should be weak in such a manner that the metric can be described:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{82}$$

where $|h_{\alpha\beta}| \ll 1$ everywhere, and $\eta_{\alpha\beta}$ is the flat Minkowski metric. What we are actually saying is that there exist coordinates in which the equation above is possible. And if this equation is true in one of these systems, then there are many other coordinate systems in which this is true. A wise choice of coordinate system is crucial.

23 Background Lorentz Transformations

The Lorentz transformations are:

$$\Lambda^{\bar{\alpha}}{}_{\beta} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = (1 - v^2)^{-\frac{1}{2}} \tag{83}$$

A Lorentz transformation is one:

$$x^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta} x^{\beta} \tag{84}$$

Although we are not in SR, let's see what happens to the metric:

$$g_{\bar{\alpha}\bar{\beta}} = \Lambda^{\mu}{}_{\bar{\alpha}} \Lambda^{\nu}{}_{\bar{\beta}} g_{\mu\nu} \tag{85}$$

$$= \Lambda^{\mu}{}_{\bar{\alpha}} \Lambda^{\nu}{}_{\bar{\beta}} \eta_{\mu\nu} + \Lambda^{\mu}{}_{\bar{\alpha}} \Lambda^{\nu}{}_{\bar{\beta}} h_{\mu\nu} \tag{86}$$

But by definition of Lorentz transformations:

$$\Lambda^{\mu}{}_{\bar{\alpha}} \Lambda^{\nu}{}_{\bar{\beta}} \eta_{\mu\nu} = \eta_{\bar{\alpha}\bar{\beta}} \tag{87}$$

So:

$$g_{\bar{\alpha}\bar{\beta}} = \eta_{\bar{\alpha}\bar{\beta}} + h_{\bar{\alpha}\bar{\beta}} \tag{88}$$

where:

$$h_{\bar{\alpha}\bar{\beta}} = \Lambda^{\mu}{}_{\bar{\alpha}} \Lambda^{\nu}{}_{\bar{\beta}} h_{\mu\nu} \tag{89}$$

which show that $h_{\mu\nu}$ transforms as if a tensor in SR itself. This property of the slightly "curved" or modified Minkowski will make it easier the calculations. All physical fields, including the Riemman tensor will be written just in terms of it.

24 The Newtonian Field

We can use this approximation and think that $h_{\mu\nu}$ is

$$h_{\mu\nu} = \begin{pmatrix} -2\phi(\vec{x}) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{90}$$

So that

$$g_{\mu\nu} = \begin{pmatrix} -(1+2\phi(\vec{x})) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{91}$$

The metric in Cartesian coordinates can be written then:

$$ds^2 = -(1+2\phi(\vec{x}))dt^2 + dx^2 + dy^2 + dz^2 \tag{92}$$

One thing that we can do is see how a particle initially at rest moves in these coordinates. In the geodesic equation,

$$\frac{d}{ds}\left(\frac{dx^\alpha}{ds}\right) + \mathit{\Gamma}^\alpha{}_{\mu\beta}\frac{dx^\mu}{ds}\frac{dx^\beta}{ds} = 0 \tag{93}$$

We can assume that for this particle, at rest at the origin, the world line is $x^\alpha(\tau) = (\tau, 0, 0, 0)$ and then $v^\alpha = \dot{x}^\alpha = \frac{dx}{d\tau} = (1, 0, 0, 0)$. This means that the only surviving term in eq (1.12) is

$$\ddot{x}^\alpha + \mathit{\Gamma}^\alpha{}_{00} = 0 \tag{94}$$

(The only surviving $\frac{dx^\mu}{ds} = \frac{dx^0}{d\tau} = 1$, all others are zero). This is essentially

$$\mathit{\Gamma}^\alpha{}_{00} = \frac{1}{2}g^{\alpha\beta}(-\partial_\beta g_{00}) = -\eta^{\alpha\beta}(\partial_\beta\phi) \tag{95}$$

Where we worked to first order in the metric (this is the key method in the weak field limit -$terms O(\phi^2)$ are deemed small enough to be neglected-). Notice then that equation (1.13) becomes

$$\ddot{\vec{x}} = -\nabla\phi. \tag{96}$$

These are Newton's equations!

25 Newton's force

Let's assume that the sources is given by an energy momentum tensor also static of the form

$$T_{\mu\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{97}$$

ρ is the matter density. We can insert this energy momentum tensor in Einstein's equations:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = 8\pi G T^{\mu\nu} \tag{98}$$

It is not that hard to show that the equation reduces to:

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = 8\pi G\rho \tag{99}$$

For a mass M concentrated at the origin this gives $\phi = -\frac{GM}{r}$. So in this approximation the spacetime geometry is given by the metric:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{100}$$

On the surface of the Earth the Newtonian field is due to the mass of the Earth $M = M_E$ at its radius $r = r_E$ gives a correction to the "Euclidean" metric $ds^2 = dx^2 + dy^2 + dz^2$,

$$\frac{2GM_E}{c^2 r_E} = \frac{2 \times 6.67 \times 10^{-8} cm^3/gs^2 \times 5.972 \times 10^{24} kg}{(3 \times 10^8 m/s)^2 \times 6,371 km} \sim 1.3 \times 10^{-9} \tag{101}$$

The correction is only of the order of one part in a billion. But this is enough to bend the path of free particles into parabolas.