INTRODUCTION TO GENERAL RELATIVITY AND GRAVITATION - 2025

HOMEWORK 9

Exercise 1.

In mathematics, a Killing vector field (often called a Killing field), named after Wilhelm Killing, is a vector field on a Riemannian manifold (or pseudo-Riemannian manifold) that preserves the metric.

A Killing vector is then a vector $\vec{X}$ that satisfies the following condition when apply to a metric

$$\mathcal{L}_{\vec{X}} g_{ab} = 0 \tag{1}$$

where $\mathcal{L}_{\vec{X}}$ is the Lie derivative respect to the vector field $\vec{X}$ and are defined

$$\begin{aligned} \mathcal{L}_{\vec{V}} g_{\mu\nu} &= V^\sigma \nabla_\sigma g_{\mu\nu} + (\nabla_\mu V^\lambda) g_{\lambda\nu} + (\nabla_\nu V^\lambda) g_{\mu\lambda} && (2) \\ &= \nabla_\mu V_\nu + \nabla_\nu V_\mu, && (3) \end{aligned}$$

where ∇_μ is the covariant derivative respect to x^μ. The Killing equations are then the following

$$X^c g_{ab,c} + g_{ac} X^c{}_{,b} + g_{bc} X^c{}_{,a} = 0 \tag{4}$$

Spherical coordinates are defined as:

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta. \tag{5}$$

If a metric is spherically symmetric then the generators of rotations are its Killing vectors. In these coordinates the generators are:

$$\begin{aligned} \underset{[xy]}{J} &= \frac{\partial}{\partial\phi} \\ \underset{[yz]}{J} &= \sin\phi \frac{\partial}{\partial\theta} + \cos\phi \cot\theta \frac{\partial}{\partial\phi} \\ \underset{[xz]}{J} &= \cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi} \end{aligned} \tag{6}$$

Write all the Killing equations for a generic symmetric metric.

Hint:

Let's for the sake of clear notation have $x^0 = t$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$. Then equation (4) for example for the Killing vector $\underset{[xy]}{J} = \frac{\partial}{\partial\phi}$ becomes

$$g_{ab,3} + g_{a3} X^3{}_{,b} + g_{b3} X^3{}_{,a} = 0 \tag{7}$$

But because X^3 for the Killing vector $\underset{[xy]}{J} = \frac{\partial}{\partial \phi}$ is 1 $g_{a3} = 0$ for any $a \neq 0$ and $X^3_{,a} = 0$ for any $a \neq 0$ so the equation reduces to:

$$g_{ab,3} = \frac{\partial g_{ab}}{\partial \phi} = 0 \tag{8}$$

Which show that the metric does not depend on the $x^3 = \phi$ coordinate. ϕ is called an ignorable coordinate.
Then working out the remaining Killing equations show that the metric solution of them has this form:

$$ds^2 = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 d\Omega^2. \tag{9}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{10}$$

and

$$g^{\alpha\beta} = \begin{pmatrix} -e^{-\nu} & 0 & 0 & 0 \\ 0 & e^{-\lambda} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix} \tag{11}$$

Exercise 2.

Show using metric (9) that the Einstein's equations become, with all other Einstein tensor components identically zero:

$$G_0{}^0 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{12}$$

$$G_0{}^1 = -e^{-\lambda} r^{-1} \dot{\lambda} = -e^{\lambda-\nu} G_1{}^0, \tag{13}$$

$$G_1{}^1 = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{14}$$

$$G_2{}^2 = G_3{}^3 = \frac{1}{2} e^{-\lambda}\left(\frac{\nu'\lambda'}{2} + \frac{\lambda'}{r} - \frac{\nu'}{r} - \frac{\nu'^2}{2} - \nu''\right)$$
$$+ \frac{1}{2} e^{-\nu}\left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2}\right). \tag{15}$$

where $\dot{}$ and $'$ denote derivatives respect to t and r respectively.

Exercise 3.

Show that the Schwarzschild metric

$$\begin{aligned} ds^2 = -(1-2m/r)dt^2 + (1-2m/r)^{-1}dr^2 \\ +r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \end{aligned} \tag{16}$$

satisfies Einstein's equations (10)-(13). Hint: use Mathematica notebooks presented in class to show that it is a solution.

Exercise 4.

Time independence of the Schwarzschild metric means that the energy $-p_0$ is a constant of the trajectory. Photons in the Schwarzschild metric have energy E:

$$E = -p_0 \tag{17}$$

Independence of the angle ϕ implies angular momentum is also constant $L = p_\phi$ and independence of θ implies motion is confined to a plane. We choose it to be the equatorial one ($\theta = \pi/2$) and then $p_\theta = 0$. The other components of the momentum are:

$$p^0 = g^{00}p_0 = m\left(1-\frac{2M}{r}\right)^{-1}E, \tag{18}$$

$$p^r = dr/d\lambda, \tag{19}$$

$$p^\phi = d\phi/d\lambda = L/r^2 \tag{20}$$

(a) Show, being a photon $\vec{p}\cdot\vec{p} = -m^2 = 0$ that this leads to:

$$-E^2\left(1-\frac{2M}{r}\right)^{-1} + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2} = 0 \tag{21}$$

(b) Show that $\left(\frac{dr}{d\lambda}\right)^2$ can be written as:

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - V^2(r) \tag{22}$$

where

$$V^2(r) = \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2} \tag{23}$$

(c) Make a sketch plot of (21) as a function of r.

Exercise 5.

Deflection of light

(a) If we calculate the trajectory of a photon in the Schwarzschild metric assuming M/r is small along the trajectory show using equations (18) and (20) that the trajectory $\frac{d\phi}{dr}$ is given by

$$\frac{d\phi}{dr} = \pm\frac{1}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-1/2} \tag{24}$$

where we define $b = L/E$ as the impact parameter.

(b) An incoming photon will obey the equation

$$\frac{d\phi}{du} = \left(\frac{1}{b^2} - u^2 + 2Mu^3\right)^{-1/2} \tag{25}$$

where $u = 1/r$ Show that neglecting the term u^3 the effect of M disappear and the solution is a straight line (the newtonian result).

(c) If we assume $Mu \ll 1$ but no totally negligible, defining $y = u(1 - Mu)$, show that the equation becomes

$$\frac{d\phi}{dy} = \frac{(1 + 2My)}{(1/b^2 - y^2)^{1/2}} + O(M^2u^2) \tag{26}$$

(d) Integrate to show that the solution to first order is:

$$\phi = \phi_0 + \frac{2M}{b} + \arcsin(by) - 2M\left(\frac{1}{b^2} - y^2\right)^{1/2} \tag{27}$$

If the initial trajectory starts at $r = \infty$ what is the value of y?

(e) What is the value of y when r reaches its smallest value?

Hint: Solve for eq (20) using (21) assuming $dr/d\lambda = 0$ and that $Mu \ll 1$

(f) What is the value of ϕ for that y?

(g) Notice that the photon will go in its way to the observer twice through the closest r to the star. The first time it will have the value from (f). As it keeps moving it will pass -farther ahead- through a point that is at the same distance of the source. Show that the new value would be twice the angle accumulating an excess angle of $\Delta\phi = 4M/b$ respect to the straight line trajectory. (you could look at Figure 11.5 from Schutz book for illustration).

Exercise 6.

Our Sun has an equatorial rotation velocity of about 2 km/s. (a) Estimate its angular momentum, on the assumption that the rotation is rigid (uniform angular velocity) and the Sun is of uniform density. As the true angular velocity is likely to increase inwards, this is a lower limit on the Sun's angular momentum.

(b) If the Sun were to collapse to neutron-star size (say 10 km radius), conserving both mass and total angular momentum, what would its angular velocity of rigid rotation be? In nonrelativistic language, would the corresponding centrifugal force exceed the Newtonian gravitational force on the equator?

(c) A neutron star of $1M_{\odot}$ and radius $10km$ rotates 30 times per second (typical of young pulsars). Again in Newtonian language, what is the ratio of centrifugal to gravitational force on the equator? In this sense the star is slowly rotating.

(d) Suppose a main-sequence star of $1M_{\odot}$ has a dipole magnetic field with typical strength 1 Gauss in the equatorial plane. Assuming flux conservation in this plane, what field strength should we expect if the star collapses to radius of 10 km? (The Crab pulsar's field is of the order of 10^{11} Gauss.)