INTRODUCTION TO GENERAL RELATIVITY AND GRAVITATION

HOMEWORK 6

Exercise 1.

A valid metric for the torus with $0 < u, v < 2\pi$ is:

$$ds^2 = (c + a\cos v)^2 du^2 + a^2 \sin^2(v) dv^2 \tag{1}$$

We can change to coordinates

$$x = (c + a\cos v)\cos u \tag{2}$$

$$y = (c + a\cos v)\sin u \tag{3}$$

Calculate ds^2 in terms of the x, y coordinates.

Exercise 2.

The metric on the 2-sphere surface is $\mathbf{S}^2$:

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4}$$

Calculate the connection coefficients. Which ones are non zero ? What are the values of them. What are the non zero components of the Riemann tensor? calculate them and calculate also the Ricci tensor. Finally calculate the Ricci scalar and show it's value is

$$R = \frac{2}{a^2}. \tag{5}$$

proving that it is indeed a curved surface.

Exercise 3.

Starting with the metric:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{6}$$

Calculate explicitly the metric:

$$g_{\bar{\alpha}\bar{\beta}} = \Lambda^{\mu}{}_{\bar{\alpha}}\Lambda^{\nu}{}_{\bar{\beta}}g_{\mu\nu} \tag{7}$$

where

$$\Lambda^{\bar{\alpha}}{}_{\beta} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \;\; \gamma = (1 - v^2)^{-\frac{1}{2}} \tag{8}$$

is the Lorentz transformations from a system x^α to another $x^{\bar{\alpha}}$ moving at speed v relative to the first one.

Exercise 4.
Show that if we start with a metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{9}$$

and make a transformation

$$x^{\alpha'} = x^\alpha + \xi^\alpha(x^\beta), \tag{10}$$

where ξ^α is small in the sense that $|\xi^\alpha{}_{,\beta}| \ll 1$ To first order this gives:

$$g_{\alpha'\beta'} = \eta_{\alpha\beta} + h_{\alpha\beta} - \xi_{\alpha,\beta} + \xi_{\beta,\alpha}, \tag{11}$$

Exercise 5.

Calculate the Riemann tensor for metric (9) and show that it gives:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}) \tag{12}$$

Hint: remember that you need to keep first order terms in $h_{\alpha\beta}$ only.

Exercise 6.
Using

$$h^\mu_\beta := \eta^{\mu\alpha} h_{\alpha\beta}, \tag{13}$$

$$h^{\mu\nu} := \eta^{\nu\beta} h^\mu_\beta, \tag{14}$$

the trace:

$$h := h^\alpha_\alpha \tag{15}$$

and another tensor called the ”trace reverse” of $h_{\alpha\beta}$

$$\bar{h}^{\alpha\beta} := h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} h. \tag{16}$$

Prove that the trace of $\bar{h}^{\alpha\beta}$ is:

$$\bar{h} := \bar{h}^\alpha_\alpha = h^{\alpha\beta}\eta_{\beta\alpha} - \frac{1}{2}\eta^{\alpha\beta}\eta_{\beta\alpha} h = h - \frac{4}{2}h = -h \tag{17}$$

Using that the inverse of (16) is the same equation:

$$h^{\alpha\beta} = \bar{h}^{\alpha_\beta} - \frac{1}{2}\eta^{\alpha\beta}\bar{h}. \tag{18}$$

Show that the Einstein tensor becomes:

$$G_{\alpha\beta} = -\frac{1}{2}[\bar{h}_{\alpha\beta,\mu}{}^{,\mu} + \eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{,\mu\nu} - \bar{h}_{\alpha\mu,\beta}{}^{,\mu} \tag{19}$$

$$-\bar{h}_{\beta\mu,\alpha}{}^{,\mu} + O(h^2_{\alpha\beta})]. \tag{20}$$

If we require:

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0. \tag{21}$$

so called Lorentz gauge. Show then that the vacuum Einstein's equation in this gauge become

$$G^{\alpha\beta} = -\frac{1}{2}\Box\bar{h}^{\alpha\beta} \tag{22}$$

And consequently the weak field Einstein equations become:

$$\Box\bar{h}^{\mu\nu} = 0 \tag{23}$$

where the box operator of a function is defined $\Box f$ of a function is defined :

$$\Box f = f^{,\mu}{}_{,\mu} = \left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) f. \tag{24}$$