**INTRODUCTION TO GENERAL RELATIVITY AND GRAVITATION**

HOMEWORK 4

2025

Exercise 1.

Explain why a uniform external gravitational field would raise no tides on Earth.

Exercise 2.

Given in the frame $\mathcal{O}$ the vectors $\vec{A} = (2,1,1,0)$, $\vec{B} = (1,2,0,0)$ $\vec{C} = (0,0,1,1)$ and $\vec{D} = (-3,2,0,0)$.
(a) find the components of $\tilde{p}$ if
$\tilde{p}(\vec{A}) = 1$, $\tilde{p}(\vec{B}) = -1$, $\tilde{p}(\vec{C}) = -1$ and $\tilde{p}(\vec{D}) = 0$;
(b) find the value of $\tilde{p}(\vec{E})$ for $\vec{E} = (1,1,0,0)$

Exercise 3.

(a) Given the components of a tensor $M^{\alpha\beta}$ as the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

find:
(i) the components of the symmetric tensor $M^{(\alpha\beta)}$ and the antisymmetric tensor $M^{[\alpha\beta]}$;
(ii) the components of $M^{\alpha}{}_{\beta}$;
(iii) the components of $M_{\alpha}{}^{\beta}$;
(iv) the components of $M_{\alpha}\beta$.
(b) For the tensor whose components are $M^{\alpha}{}_{\beta}$, does it make sense to speak of its symmetric and antisymmetric parts? If so, define them. If not, say why.

Exercise 4.

(a) Show that the coordinate transformation $(x, y) \to (\zeta, \eta)$ with $\zeta = x$ and $\eta = 1$ violates

$$det \begin{pmatrix} \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} \neq 0$$

(b) Are the following coordinates transformations good ones? Compute the jacobian and list any points where the transformations fail.
(i) $\psi = (x^2 + y^2)^{1/2}, \quad \eta = \arctan(y/x)$,
(ii) $\psi = \ln(x), \quad \eta = y$,

Exercise 5.

Calculate all elements of the transformation matrices $\Lambda^{\alpha'}{}_{\beta}$ and $\Lambda^{\mu}{}_{\nu'}$ for the transformation from Cartesian $(x, y)$ - the unprimed indices - to polar $(r, \theta)$ - the primed indices.

Exercise 6.

Let $f = x^2 + y^2 + 2xy$, and in Cartesian coordinates $\vec{V} = (x^2 + 3y, y^2 + 3x)$, and $\vec{W} = (1, 1)$.
(a) Compute $f$ as a function of $r$ and $\theta$, and find the components of $\vec{V}$ and $\vec{W}$ on the polar basis, expressing them as functions of $r$ and $\theta$.
(b) Find the components of $f$ in Cartesian coordinates and obtain them in polars (i) by direct calculation in polars, and (ii) by transforming components from Cartesian.
(c) (i) Use the metric tensor in polar coordinates to find the polar components of the one-forms $\tilde{V}$ and $\tilde{W}$ associated with $\vec{V}$ and $\vec{W}$. (ii) Obtain the polar components of $\tilde{V}$ and $\tilde{W}$ by transformation of their Cartesian components.

Exercise 7.

Use the results of Exercise 5) and 6) considering again $\vec{V} = (x^2 + 3y, y^2 + 3x)$ to compute:
a) $V^{\alpha}{}_{,\beta}$ in Cartesian coordinates.
b) The transformation $\Lambda^{\mu'}{}_{\alpha}\Lambda^{\beta}{}_{\mu'}V^{\alpha}{}_{,\beta}$ to polar coordinates.
c) the components $V^{\mu'}{}_{;\nu'}$ directly in polar coordinates using the Christoffel symbols (38) to (41) from Lesson notes 4.
d) the divergence $V^{\alpha}{}_{,\alpha}$ using a).
e) the divergence $V^{\mu'}{}_{;\mu'}$ using b) or c).
f) the divergence $V^{\mu'}{}_{;\mu'}$ using equation (58) from Lesson 4 notes.

Exercise 8.

For the vector whose polar components are $(V^r = 1, V^{\theta} = 0)$ compute in polar coordinates all the components of the second covariant derivative $V^{\alpha}{}_{;\mu;\nu}$ (Hint: treat the first covariant derivative as a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor.