INTRODUCTION TO GENERAL RELATIVITY AND GRAVITATION

HOMEWORK 3

Exercise 1.

Show that the number density of dust measured by an arbitrary observer whose four-velocity is $\vec{U}_{obs}$ is $\vec{N} \cdot \vec{U}_{obs}$.

Exercise 2.

Show that ${T^{\alpha\beta}}_{,\beta} = 0$, when α is any spatial index, is just Newton's second law.

Exercise 3.

Show that in a momentarily co-moving reference frame

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{1}$$

can be written like this:

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu}, \tag{2}$$

Exercise 4.

In an inertial frame $\mathcal{O}$ calculate the components of the stress-energy tensors of the following system:
(a) A group of particles all moving with the same velocity $\vec{v} = \beta\vec{e}_x$, as seen in $\mathcal{O}$. Let the rest-mass density of these particles be ρ_0, as measured in their comoving frame. Assume a sufficiently high density of particles to enable treating them as a continuum.

Exercise 5.

Many physical systems may be idealized as collections of non-colliding particles (for example, black-body radiation, rarified plasmas, galaxies, and globular clusters). By assuming that such a system has a random distribution of velocities at every point, with no bias in any direction in the MCRF, prove that the stress-energy tensor is that of a perfect fluid. If all particles have the same speed u and mass m, express p and ρ as functions of m, u, and n. Show that a photon gas has

$p = \frac{1}{3}\rho$.

Exercise 6.

Use the identity $T^{\mu\nu}{}_{,\nu} = 0$ to prove the following results for a bounded system (i.e. a system for which $T^{\mu\nu} = 0$ outside a bounded region of space):
(a)$\frac{\partial}{\partial t} \int T^{0\alpha} d^3x = 0$ (conservation of energy and momentum).

Exercise 7.

Defining the Electromagnetic tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} = -F_{\nu\mu} \tag{3}$$

Prove that Maxwell's equation in component notation:

$$\epsilon^{ijk}\partial_j B_k - \partial_0 E^i = J^i \tag{4}$$
$$\partial_i E^i = J^0 \tag{5}$$
$$\epsilon^{ijk}\partial_j E_k + \partial_0 B^i = 0 \tag{6}$$
$$\partial_i B^i = 0 \tag{7}$$

can be written in terms of (3):

$$\partial_j F^{ij} - \partial_0 F^{0i} = J^i \tag{8}$$
$$\partial_i F^{0i} = J^0 \tag{9}$$

and then combined:

$$\partial_\mu F^{\nu\mu} = J^\nu \tag{10}$$

and

$$\partial_{[\mu} F_{\nu\lambda]} = 0 \tag{11}$$

Exercise 8.

Use the identity $T^{\mu\nu}{}_{,\nu} = 0$ to prove the following results for a bounded system (i.e. a system for which $T^{\mu\nu} = 0$ outside a bounded region of space):
(b) $\frac{\partial^2}{\partial t^2} \int T^{00} x^i x^j d^3x = 2 \int T^{ij} d^3x$ (tensor virial theorem).
(c) $\frac{\partial^2}{\partial t^2} \int T^{00} (x^i x_i)^2 d^3x = 4 \int T^i_i x^j x_j d^3x + \int T^{ij} x_i x_j d^3x$