

# Introduction to GR-2025

## Homework 2

### 1 Exercise 1

Given the numbers:

$\{A^0 = 5, A^1 = 0, A^2 = -1, A^3 = -6\}, \{B_0 = 0, B_1 = -2, B_2 = 4, B_3 = 0\}$ , and,

$$C = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 5 & -2 & -2 & 0 \\ 4 & 5 & 2 & -2 \\ -1 & -1 & -3 & 0 \end{pmatrix} \text{ find:}$$

- (a)  $A^\alpha B_\alpha$ ;
- (b)  $A^\alpha C_{\alpha\beta}$  for all  $\beta$ ;
- (c)  $A^\gamma C_{\gamma\sigma}$  for all  $\sigma$ ;
- (d)  $A^\nu C_{\mu\nu}$  for all  $\mu$ ;
- (e)  $A^\alpha B_\beta$  for all  $\alpha, \beta$ ;
- (f)  $A^i B_i$ ;
- (g)  $A^j B_k$  for all  $j, k$ ;

### 2 Exercise 2

Identify the free and dummy indices in the following equations and change them into equivalent expressions with different indices. How many different equations does each expression represent?

- (a)  $A^\alpha B_\alpha = 5$ ;
- (b)  $A^\mu = \Lambda_\nu^\mu A^\nu$ ;
- (c)  $T^{\alpha\mu\lambda} A_\mu C_\lambda{}^\gamma = D^{\gamma\alpha}$ ;
- (d)  $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = G_{\mu\nu}$ ;

### 3 Exercise 3

A collection of vectors  $\{\vec{a}, \vec{b}, \vec{c}, \vec{d}\}$  is said to be linearly independent if no linear combination of them is zero except the trivial one,  $0\vec{a} + 0\vec{b} + 0\vec{c} + 0\vec{d} = 0$ .

- (a) Show that the basis vectors  $\vec{e}_0 = (1, 0, 0, 0), \vec{e}_1 = (0, 1, 0, 0), \vec{e}_2 = (0, 0, 1, 0), \vec{e}_3 = (0, 0, 0, 1)$  are linearly independent.
- (b) Is this set of basis vectors  $\{\vec{a}, \vec{b}, \vec{c}, 5\vec{a} + 3\vec{b} - 2\vec{c}\}$  LI?

#### **4 Exercise 4**

- (a) Prove that the zero vector  $(0, 0, 0, 0)$  has these same components in all reference frames.
- (b) Use (a) to prove that if two vectors have equal components in one frame, they have equal components in all frames.

#### **5 Exercise 5**

- (a) Show that the sum of any two orthogonal spacelike vectors is spacelike.
- (b) Show that a timelike vector and a null vector cannot be orthogonal.

#### **6 Exercise 6**

Write down the change of coordinates from Cartesian  $(x, y, z)$  to spherical coordinates  $(r, \theta, \phi)$ . Obtain the transformation matrices from one to the other. Write  $(1, 0, 0)$  and  $(0, 1, 0)$  and  $(0, 0, 1)$  as vector operators in both coordinate systems. Show how the gradient in the direction orthogonal to spheres of constant radius transform from one system to the other.