Homework 13

Introduction to General Relativity and Gravitation - 2026

Exercise 1

Suppose the present value of the Hubble constant is 72(km/s/Mpc) and that the current density of the universe is the critical density. A photon is emitted form our galaxy now. What is the redshift of this photon when it is received in another galaxy 10 billion years in the future, assuming the universe continues to be matter dominated.

Exercise 2

As the universe expands the horizon grows. Estimate the time it has to grow for one new galaxy to come within the horizon, assuming the universe was matter dominated throughout its whole history.

Exercise 3

(Radiation Dominated FRW Models.) Solve the Friedman equation to exhibit the scale factor as a function of time for FRW models that are radiation dominated from start to finish. Express your answers in terms of H_0 and $\Omega = \Omega_R$.

Exercise 4

(de Sitter Space) Solve the Friedman equation for the scale actor as a function of time for closed FRW models that have only vacuum energy ρ_Λ. Do these models have an initial big-bang singularity?

Exercise 5

Find a closed form solution to the dynamical equation for the flat FRW models in the case when there is no radiation $\Omega_R = 0$ but both vacuum energy and matter are present. Express your answer in terms of H_0, Ω_M, and $\Omega_\Lambda = 1 - \Omega_M$. How large would Ω_Λ, have to be for the universe to be accelerating ($\ddot{a} > 0$) at the present time? Find an explicit expression for the age of the universe to as a function of H_0 and Ω_Λ.

Exercise 6

(The Einstein Static Universe) Consider a closed ($k = +1$) FRW model containing a matter density ρ_M, a vacuum energy density corresponding to a positive cosmological constant Λ, and no radiation. (a) Show that for a given value of Λ there is a critical value of ρ_M for which the scale factor does not change with time. Find this value. (b) What is the spatial volume of this universe in terms of Λ? (c) If ρ_M differs slightly from this value the scale factor will vary in time. Does the evolution remain close to the static universe or diverge from it? Comment: This is the Einstein static universe for which Einstein originally introduced the cosmological constant.

Exercise 7

Estimate the smallest value of the Ω_Λ, that would allow the universe to bounce at a small radius, but still reach a temperature $T \sim 10^{10}$ K such that nucleosynthesis could occur. Assume $\Omega_R = 8 \times 10^{-5}$and $\Omega_M = 0.3$.

Exercise 8

Could the observed vacuum mass-energy density in the universe be a consequence of quantum gravity? One obstacle to such an explanation is the great difference in scale between observed vacuum mass density ρ_Λ and the Planck mass density $\rho_{Pl} = c^5/\hbar G^2$ that might be expected on dimensional grounds to characterize quantum gravitational phenomena.
(a) Show that ρ_{Pl} is the correct combination of $\hbar, G$, and c with the dimensions of mass density.
(b) Evaluate the ratio ρ_Λ/ρ_{Pl}.

Exercise 9

Show that the effective distance d_{eff} of a galaxy in a spatially flat universe at redshift z can be written as

$$d_{eff} = \int_0^z \frac{dz'}{H(z')}$$

where $H(z')$ is the value of the Hubble constant at the time at when light from a galaxy with red shift z' was emitted.