Lesson 9:

Dynamical Systems

(the material in these slides follows Guillermo Abramson's

Analytical Mechanics)

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One dimensional flows

The state of a physical system in phase space satisfies a flow $(\dot{q}, \dot{p})$.

One simple model for population growth in an organism is

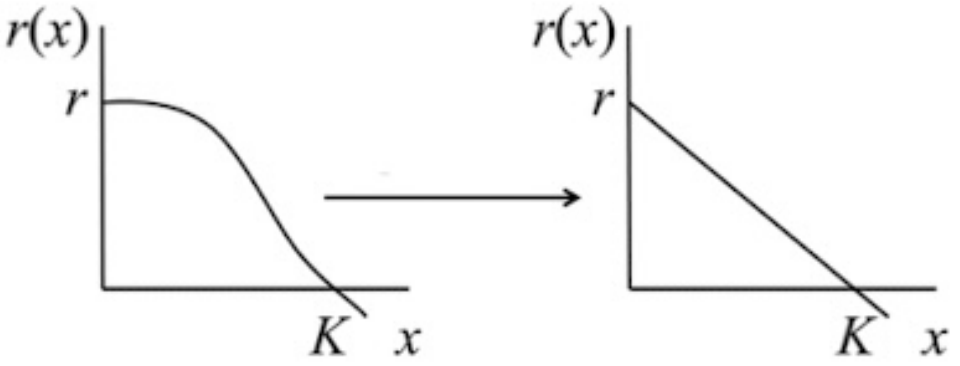
$$\dot{x} = rx \tag{1}$$

where $x(t)$ is the population and r is the reproduction rate. (Malthus model).

The solution for $r > 0$ is $x = x_0 e^{rt}$.

If resources are limited rather than considering a constant r we can think that depends on x in a decreasing capacity like in the Figure.

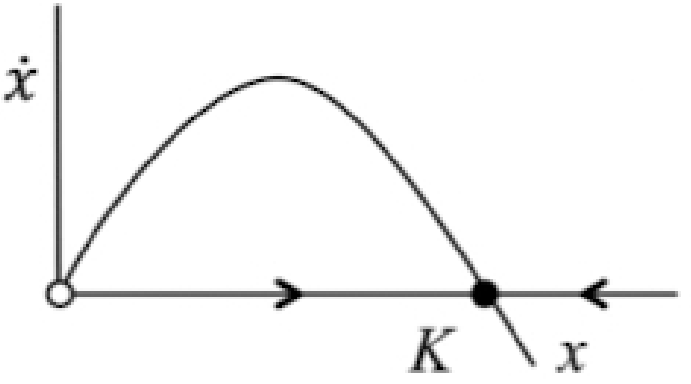
A simplification would be to consider a linearly decreasing dependency with x.



A simple non-linear model would be the logistic equation:

$$\dot{x} = rx\left(1 - \frac{x}{K}\right), \tag{2}$$

One thing we can do is to plot $\dot{x}$ vs x

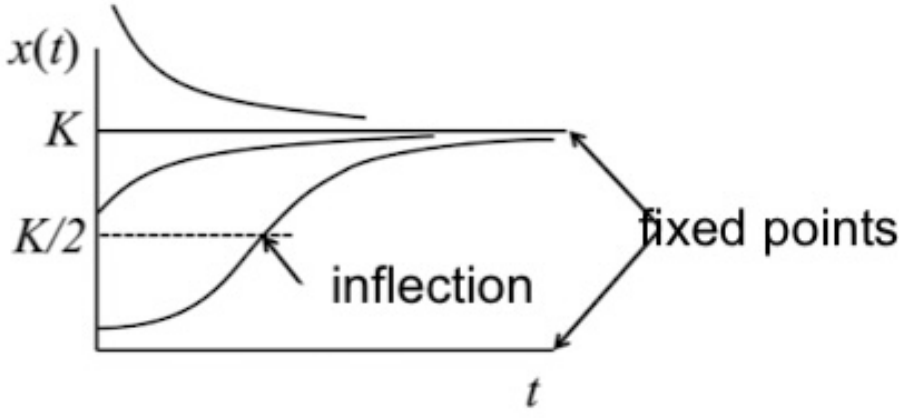


Clearly $\dot{x}$ between 0 and K is positive, and then x grows.

But when $x > K$, $\dot{x}$ is negative and x decreases.

But we are not interested in that region (population diminishes).

Let's plot x as function of possible initial conditions: 1) $x(t = 0) > K$, 2) $x(t = 0) = K$, 3) $K > x(t = 0) > K/2$. 4) $0 < x(t = 0) < K/2$.



The solution plotted on the figure corresponds to the solution:

$$x(t) = \frac{K}{1 + \left(\frac{K - x_0}{x_0}\right) e^{-rt}}, \tag{3}$$

where $x_0 = x(t = 0)$.

In the $\dot{x}$ vs x plot we see two points where $\dot{x} = 0$.

Looking at the $x(t)$ plot $x^* = 0$ seems to repel the flow and $x^* = K$ seems to attract it.

We can try to look at the stability of these points. Let's study a departure from equilibrium

$$\delta t = x(t) - x^*, \tag{4}$$

Notice that the time derivative $\dot{\delta} = \dot{x}$. We can now think of it as

$$\dot{\delta} = \dot{x} = f(x^* + \delta) \tag{5}$$

Expanding in Taylor series (respect to x)

$$\dot{\delta} = f(x^*) + \delta f'(x^*) + O(\delta^2) \tag{6}$$

To first order then and using $f(x^*) = 0$

$$\dot{\delta} = \delta f'(x^*) \tag{7}$$

Looking at $f(x)$ which is equation (2), the logistic equation, we have

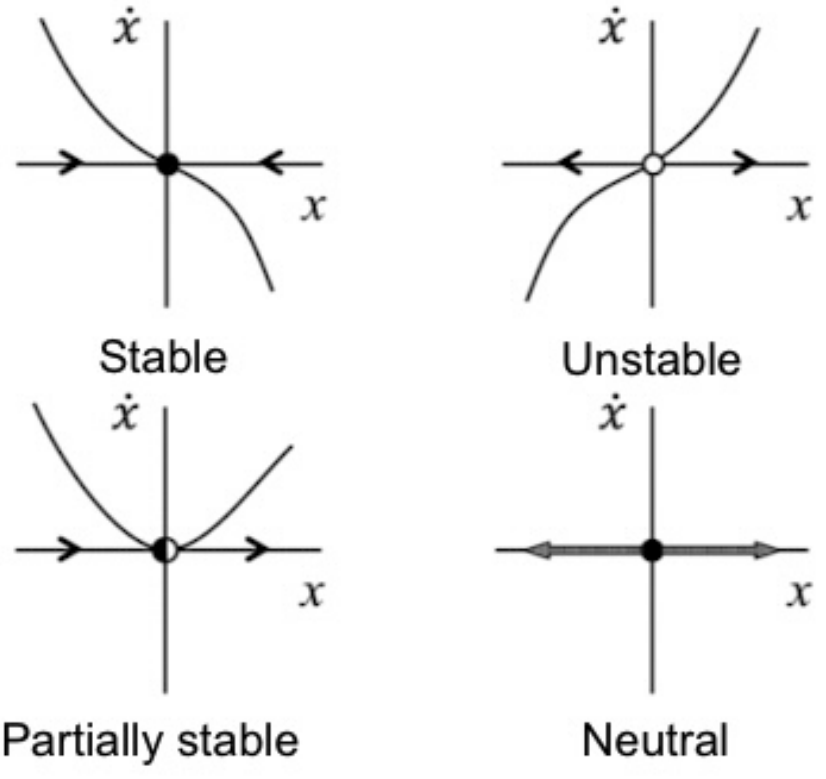
$$f' = r\left(1 - \frac{2x}{K}\right) \tag{8}$$

And $f'(x^* = 0) = r > 0$. The derivative is a growing exponential $\rightarrow$ unstable.

If $f'(x^* = K) = -r < 0$. The derivative is a decreasing exponential $\rightarrow$ stable.

Note: The stability of $x^* = K$ is different than stability in small oscillations for linear systems. In this case, trajectories are enclosed around the equilibrium point.

In this non-linear case trajectories approach equilibrium: asymptotically stable. They do not exist in conservative systems (like the ones we studied in small oscillations). They do appear in dissipative systems.



Dynamical systems in one dimension.

1D Bifurcations

1-D phase flow depends on the value of parameters.

Change in the parameters could induce changes in the fixed points.

Fixed points could be created or destroyed or change in general. Their stability could also change.

The dynamical qualitative changes are called **bifurcations**, and the values of the parameters for which change occurs are called **bifurcation points**.

Their importance resides in the modeling of transitions or instabilities that may occur in a system due to the change of a control parameter.

An example: the Tacoma bridge and the wind strength that set the bridge in oscillatory resonance breaking it.

The parameter here would be the wind strength and the transversal oscillation the variable x.

The load on a construction piece (ie. a balcony) could provide a similar example with the deflection from the vertical being the 1-dimensional variable.

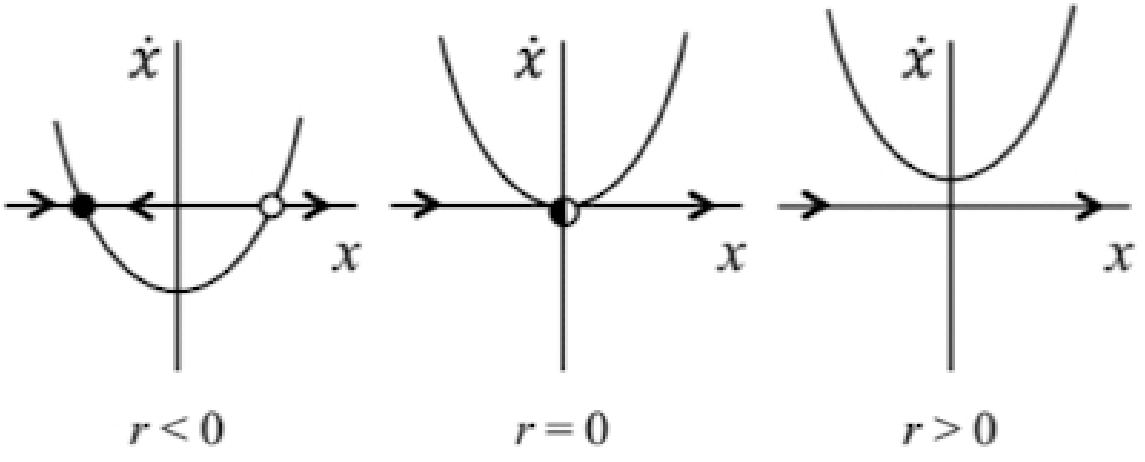
The basic mechanism for creating and destroying equilibrium is the saddle-node bifurcation:

When changing the control parameter 2 fixed stability points get closer, collide, annihilate and disappear.

A good example is:

$$\dot{x} = r + x^2. \tag{9}$$

When plotting it



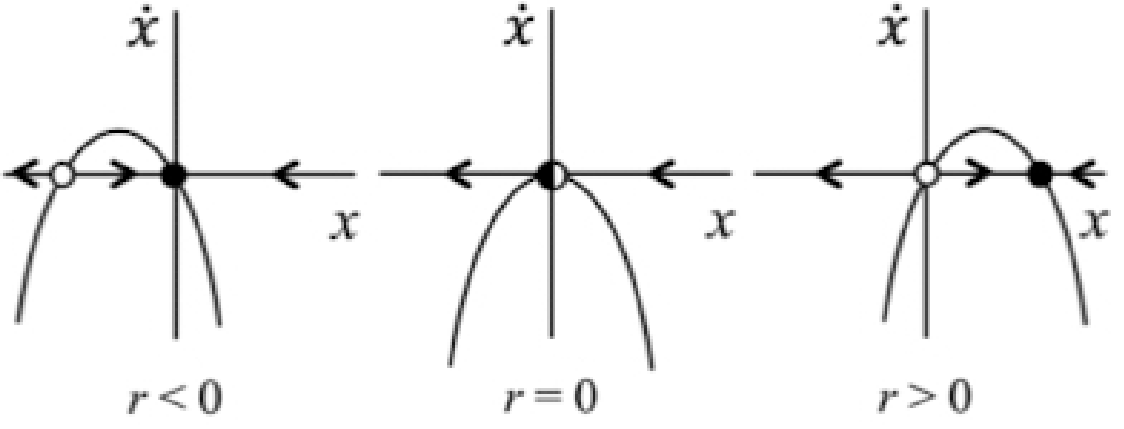
Phase plot of a saddle-node bifurcation.

The bifurcation experienced by a system represented with the logistic equation is of a type called **trans-critical bifurcation**.

A local bifurcation where two equilibrium points collide and exchange their stability as a parameter varies, without any fixed points being created or destroyed.

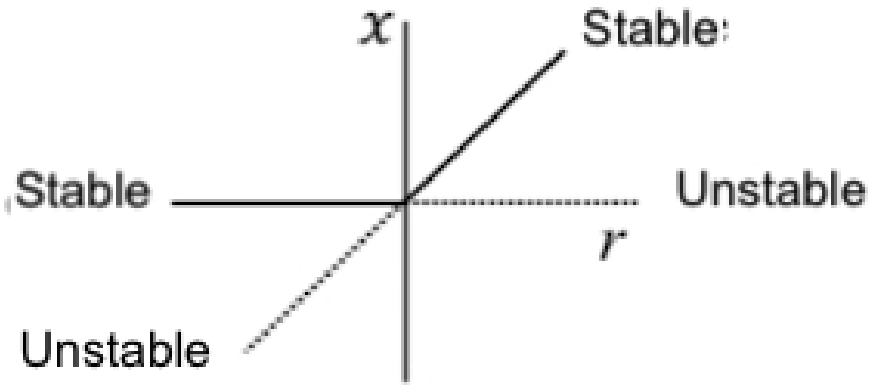
$$\dot{x} = rx - x^2. \tag{10}$$

It is similar to the logistic equation. If we plot it



Phase plot of a transcritical bifurcation.

$$\dot{x} = rx - x^2.$$

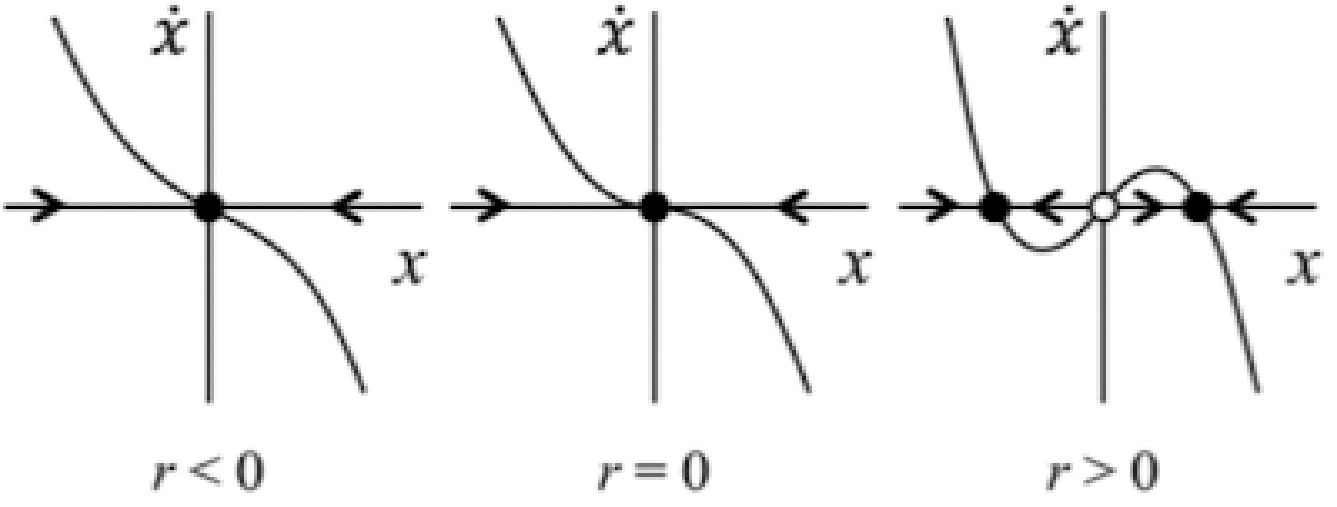


transcritical bifurcation diagram.

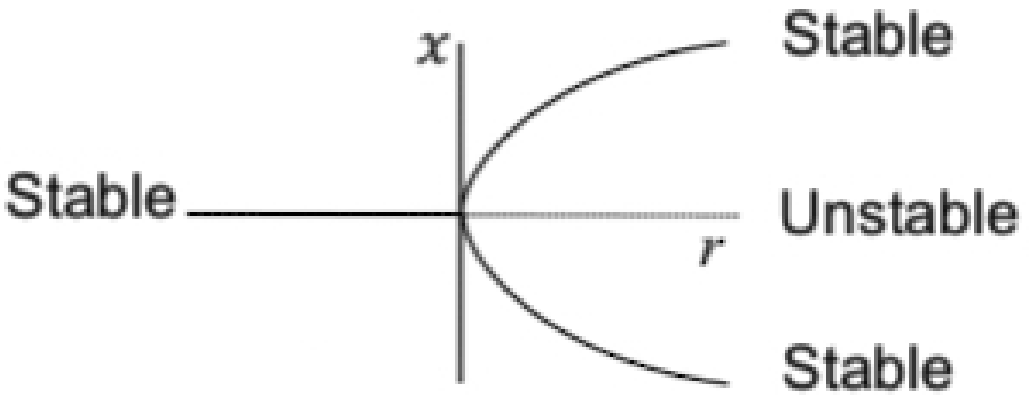
Let's add complexity. The system

$$\dot{x} = rx - x^3. \tag{11}$$

is the prototype of a pitchfork bifurcation.



pitchfork bifurcation flow.



pitchfork bifurcation diagram.

When r varies from positive to negative the fixed stable points seem to be moving into the unstable one at the origin giving rise to one stable equilibrium point.

The system is said to be bistable.

The phase plane

Let's look at a system with 2-degrees of freedom, ie. a 2-dimensional system.

$$\frac{dx(t)}{dt} = f\Big(x(t), y(t)\Big), \tag{12}$$

$$\frac{dy(t)}{dt} = g\Big(x(t), y(t)\Big), \tag{13}$$

f and g are very general functions, non lineal in general.

The solutions are curves in the plane (x, y) which define a flow.

Due to the fact that they are defined parametrically

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}, \tag{14}$$

i.e. they have a well defined derivative at each point and then they not get across, except where g/f is not well defined.

Let's analyze their equilibrium and linear stability. The system's equilibrium is defined by the values of (x, y) for which $\dot{x} = \dot{y} = 0$:

$$f\Big(x,y\Big) = 0, \qquad g\Big(x,y\Big) = 0, \tag{15}$$

where each one of them define a curve in the space (x, y). These curves are called **nullclines**.

Their intersections (where both equations are satisfied simultaneously) define the equilibrium points of the system. Let's call them x^* and y^*.

Each nullcline divides the plane phase in two regions: in one each one of $\dot{x}, \dot{y}$ have one sign and the opposite in the other region.

All we need is to see one point to find out. This makes simple to visualize the phase flow without resolving the differential equations.

Additionally as the velocity is zero along the null-clines we know that the flow cross them in the vertical or horizontal direction.

The equilibrium points where $\dot{x}, \dot{y}$ are zero the dynamics stops. These are the fixed points of the dynamical system.

The behavior of the solutions in the proximity of the fixed points (and not the points themselves) reveals the structure of the solutions.

To study the behavior of them we make the following variables substitution:

$$x(t) = x^* + \epsilon(t), \qquad y(t) = y^* + \eta(t), \quad (16)$$

Taking the derivatives and replacing in (12) and (13)

$$\begin{aligned} \dot{\epsilon(t)} &= f(x^* + \epsilon, y^* + \eta), \\ \dot{\eta} &= g(x^* + \epsilon, y^* + \eta). \end{aligned} \quad (17)$$

Expanding in power series

$$\begin{aligned} \dot{\epsilon} &= f(x^*, y^*) + \left.\frac{\partial f}{\partial x}\right|_{x^*,y^*} \epsilon + \left.\frac{\partial f}{\partial y}\right|_{x^*,y^*} \eta + O(\epsilon^2, \eta^2, \epsilon\eta), \\ \dot{\eta} &= g(x^*, y^*) + \left.\frac{\partial g}{\partial x}\right|_{x^*,y^*} \epsilon + \left.\frac{\partial g}{\partial y}\right|_{x^*,y^*} \eta + O(\epsilon^2, \eta^2, \epsilon\eta), \end{aligned} \quad (18)$$

But here $f(x^*, y^*) = g(x^*, y^*) = 0$ because x^*, y^* are equilibrium points.

If we focus on the dynamics around these points

$$\dot{\epsilon} = \left.\frac{\partial f}{\partial x}\right|_{x^*,y^*} \epsilon + \left.\frac{\partial f}{\partial y}\right|_{x^*,y^*} \eta,$$
$$\dot{\eta} = \left.\frac{\partial g}{\partial x}\right|_{x^*,y^*} \epsilon + \left.\frac{\partial g}{\partial y}\right|_{x^*,y^*} \eta, \qquad (19)$$

this is a linear first order system of differential equations. We can try a solution

$$\epsilon = c_1 e^{\lambda t}, \qquad \eta = c_2 e^{\lambda t}, \qquad (20)$$

Where we need to determine c_1, c_2, λ. Replacing in the equations

$$c_1 \lambda e^{\lambda t} = c_1 \frac{\partial f}{\partial x}\Big|_{x^*,y^*} e^{\lambda t} + c_2 \frac{\partial f}{\partial y}\Big|_{x^*,y^*} e^{\lambda t},$$
$$c_2 \lambda e^{\lambda t} = c_1 \frac{\partial g}{\partial x}\Big|_{x^*,y^*} e^{\lambda t} + c_2 \frac{\partial g}{\partial y}\Big|_{x^*,y^*} e^{\lambda t}, \quad (21)$$

And now we have to solve the algebraic system

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \quad (22)$$

The solution is given by

$$\lambda^2 - \lambda \, tr J + det J = 0 \quad (23)$$

where J is the matrix

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

tr is the trace of J and det is the determinant of J.

If there are two different eigenvalues the solutions will be superposition of two exponentials.

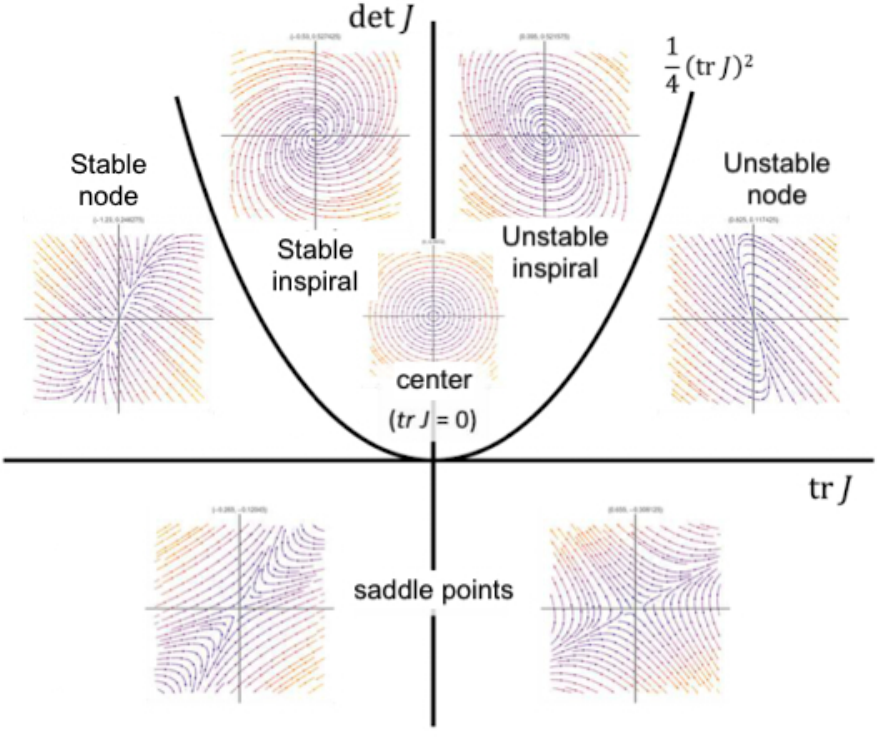
If the eigenvalues are the same then the solution will be of the shape

$$(c_1 + c_2 t)e^{\lambda t}$$

The stability of the equilibrium situation can be analyzed accordingly:

If the real part of both eigenvalues is negative, the solutions approach exponentially the fixed point: asymptotically stable.

If at least one of them is positive: the equilibrium is unstable. If the eigenvalues are complex they will be complex conjugates, and if the imaginary part is non zero there will harmonic oscillations around the equilibrium.



2-D system stability.

Example

Two herbivore species compete, let's say rabbits and sheep.

We can model their independent growth in absence of the other one:

$$\begin{aligned} \dot{x} &= x(3-x) \\ \dot{y} &= y(2-y) \end{aligned} \tag{24}$$

We are assuming rabbits have a slightly higher reproductive rate than sheep.

No let's assume an interaction when they get together: sheep being bigger displaced rabbits. And the effect is proportional to the population size... and... it's worse for rabbits:

$$\begin{aligned} \dot{x} &= x(3-x) - 2xy \equiv f(x,y) \\ \dot{y} &= y(2-y) - yx \quad \equiv g(x,y) \end{aligned} \tag{25}$$

We can plot the nullclines $\dot{x} = \dot{y} = 0$ and find 4 equilibrium points: $(0,0), (0,2), (3,0), (1,1)$.

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix} \tag{26}$$

Equilibrium $(0,0)$

$$J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \tag{27}$$

The eigenvalues are $\lambda = 3, 2$, and then $(0,0)$ is an unstable node.

Trajectories run away parallel to the eigenvector of $\lambda = 2$ which in this case is given by the y axis.

The rule is that trajectories are tangent to the slower eigen-direction.

Equilibrium $(0, 2)$

$$J = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \tag{28}$$

The eigenvalues are -1 and -2.

Both negatives and then this is a stable node.

The slower eigenvector (for -1) is $(1, -2)$, and the trajectories approach equilibrium.

Equilibrium $(3, 0)$

$$J = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \tag{29}$$

Which has eigenvalues -3 and -1.

This is then also a stable node.

In this case the slower eigenvector is in the direction $(3, -1)$, and the trajectories reach equilibrium.

Equilibrium $(1, 1)$

$$J = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \tag{30}$$

The eigenvalues are $\lambda = -1 \pm \sqrt{2}$.

This means there is one positive and one negative: a saddle point.

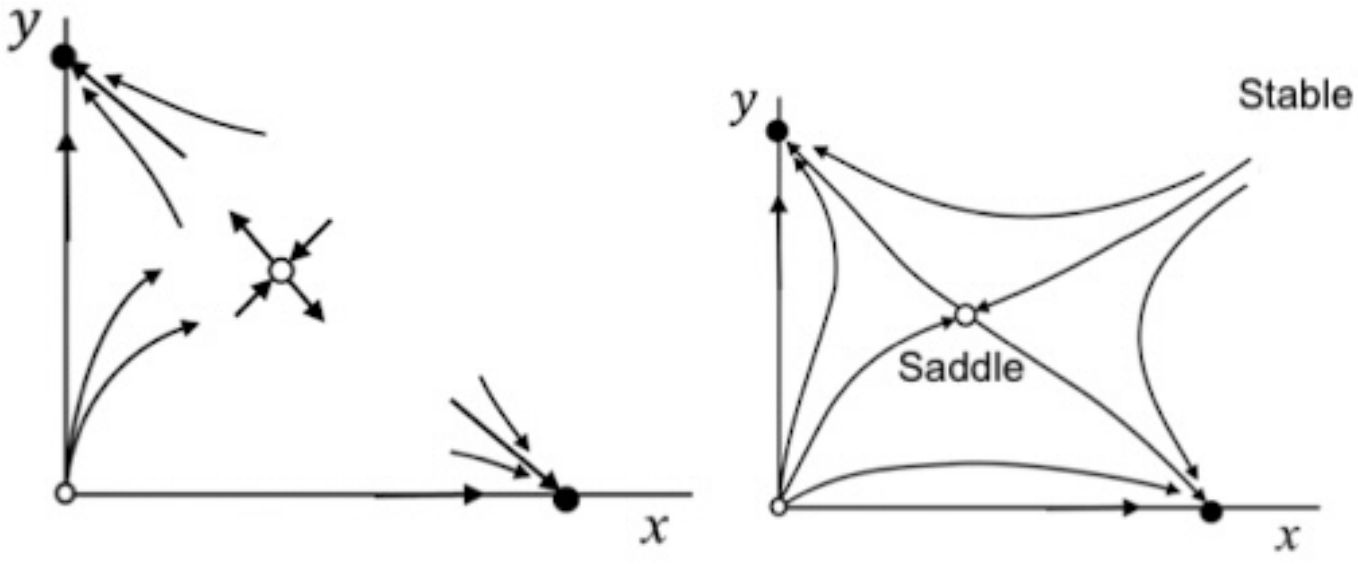
The flow near this equilibrium point has a stable direction (the one corresponding to the eigenvalue

$-1-\sqrt{2}$) and one unstable one (that of $-1+\sqrt{2}$ which it is positive).

Combining these analyses near equilibrium we can picture the linear flow.

It can be complemented with the trajectories along the axes, which when $x = 0$ gives $\dot{x} = 0$, a trajectory along the y axis and similarly along the x axis.

We can then join the trajectories filling up the rest of the phase space.



Competition model analysis.

The trajectories that leave the origin will go to one or the other equilibrium point.

One of them will go to the coexistence equilibrium.

With no other equilibrium points, no other trajectories will cross themselves.

Those from one side will reach the equilibrium point above. Those on the other side will go to equilibrium point below.

On the opposite side of $(0,0)$ there is a trajectory that comes from afar and ends at the saddle point: this is a stable equilibrium manifold. Stable but curved.

Interpretation: If the initial condition is below the stable manifold sheep will suffer extinction. If it's above, rabbits will. Coexistence is impossible.

The only special trajectory is the one that separates the two regions, the *separatrix*. But this is a condition that has zero measure in the space of initial conditions.

In Biology this dichotomy is called the principle of competitive exclusion: n species cannot coexist competing for $n-1$ limited resources.

Example: Neanderthals extinction.

Example

Volterra proposed the following model to explain the oscillations in the shoal of fish in the Adriatic Sea:

$$\begin{aligned} \dot{x} &= x(a - by), \\ \dot{y} &= y(cx - d), \end{aligned} \tag{31}$$

where x is the prey and y the predator, a, b, c, d constants.

The model assumes:

In the absence of predators, prey grows exponentially.

The effect of predators is to reduce the number of prey in a number proportional to both populations.

In the absence of prey predators disappear exponentially.

Prey contributes to the growth of predators in a manner proportional to their population.

This system can be redefine to a simpler model

$$\begin{aligned} \dot{u} &= u(1-v), \\ \dot{v} &= \alpha v(u-1), \end{aligned} \tag{32}$$

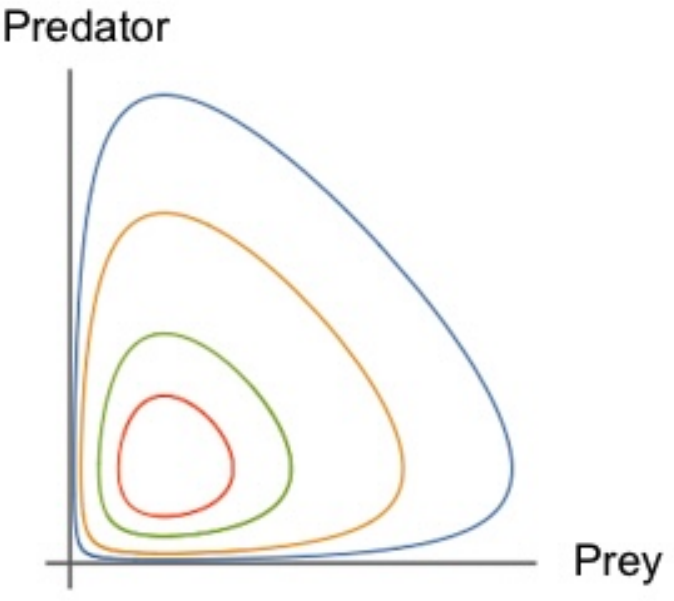
The equilibrium points are $(0,0)$ and $(1,1)$. And we get

$$J\Big|_{u=v=0} = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix} \tag{33}$$

and

$$J\Big|_{u=v=1} = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix} \tag{34}$$

The first one is a saddle point and the second one the center. The trajectories are closed curves around it.



Lotka-Volterra cycle.

For the latter $\lambda = \pm i\sqrt{\alpha}$.

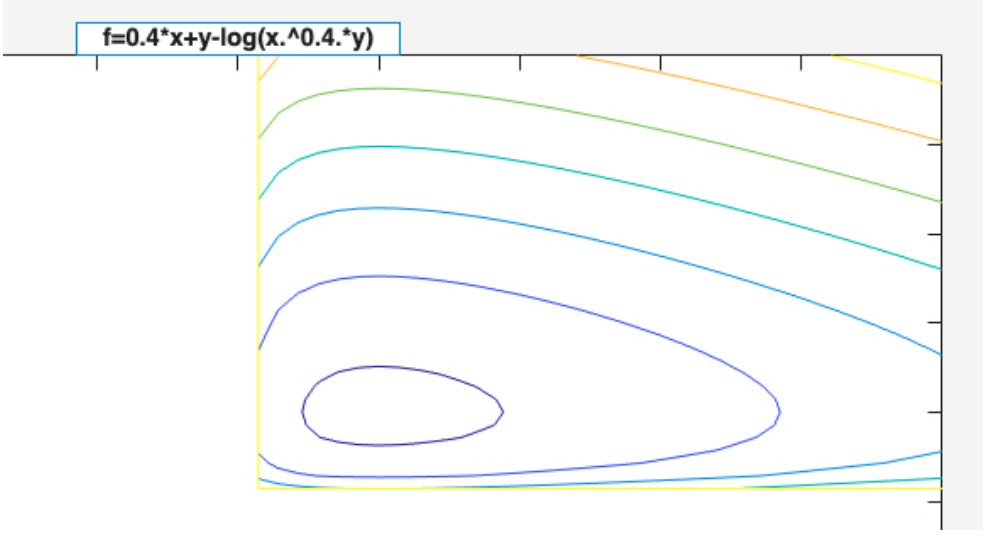
The trajectories are exponentials of the form $e^{\lambda t}$.

λ are imaginary, then the solution is a harmonic oscillator, with the period given by the imaginary part of the eigenvalues:

$$\tau = \frac{2\pi}{\sqrt{\alpha}} = 2\pi\sqrt{\frac{a}{d}}, \tag{35}$$

which depends of the ratio between the rate of growth of prey and the death of predators.

Far from equilibrium the trajectories are also cycles but not harmonics.



The curves are obtained plotting the solution $H = \alpha u + v + \ln(u^{\alpha} v)$ for (32).

Limit Cycles

Oscillations in model like the Lotka-Volterra have difficulties.

They are not structurally stable. The period depends on the amplitude of the oscillation (and then on the initial conditions).

In many physical systems a small perturbation of a cycle with a well defined frequency produces a return to the same cycle.

These are stable cycles, and they may be self-sustained. They don't need an external force to excite them.

Examples are heartbeats, circadian biological rhythms, hormone secretion, vibrations in bridges, etc.

In non-linear systems we can have closed trajectories in phase space that attract (or repel) trajectories in their neighborhood.

These orbits are called Limit cycles:

Stable (Attracting) Limit Cycle: Nearby trajectories spiral or move toward the closed loop as time approaches infinity ($t \to \infty$)).

Unstable Limit Cycle: Nearby trajectories move away from the closed loop as time approaches infinity (and approach it as $t \to -\infty$)).

Semi-Stable Limit Cycle: Attracting from one side and repelling from the other.

Example

$$\begin{aligned} \dot{r} =& r(1 - r^2), \\ \dot{\theta} =& 1, \end{aligned} \tag{36}$$

$r(t)$ and $\theta(t)$ are uncoupled.

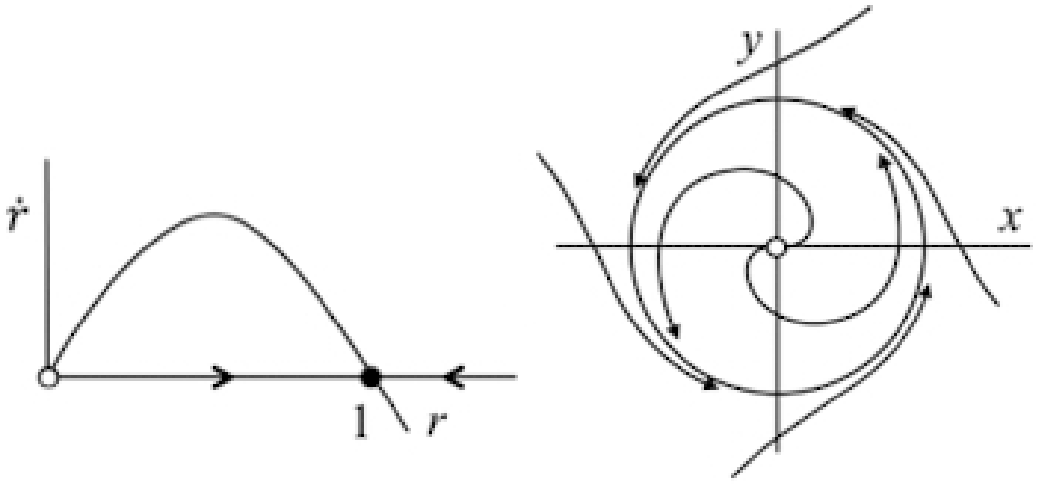
Equilibrium in the radial direction corresponds to $r(1 - r^2) = 0$ which gives $r^* = 0$ which is unstable and $r^* = 1$ which is stable.

We get that $df/dr = 1 - 3r^2$ which in $r = 0$ gives $1 > 0$ and it is an unstable point.

In $r = 1$ it gives -2 which being negative is stable. Trajectories move away from the origin and approach the circle of radius 1.

Due to the fact that the angular velocity is constant, in the circle the trajectory is harmonic periodic motion.

Inside the circle the trajectories are spirals that coil up from inside to the circle of radius 1, and from outside they tighten around it.



Left: phase space for r. Right: Phase flow.

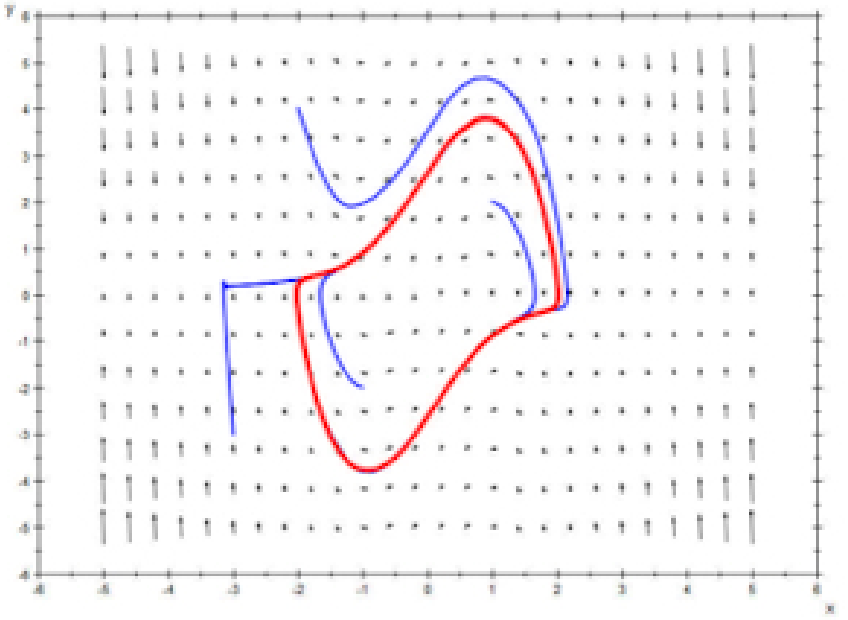
van der Pol oscillator

This is the workhorse of non-linear oscillators.

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0. \qquad (37)$$

If $|x| > 1$ oscillations are damped, but low amplitude oscillations have a "negative damping" and are amplified.

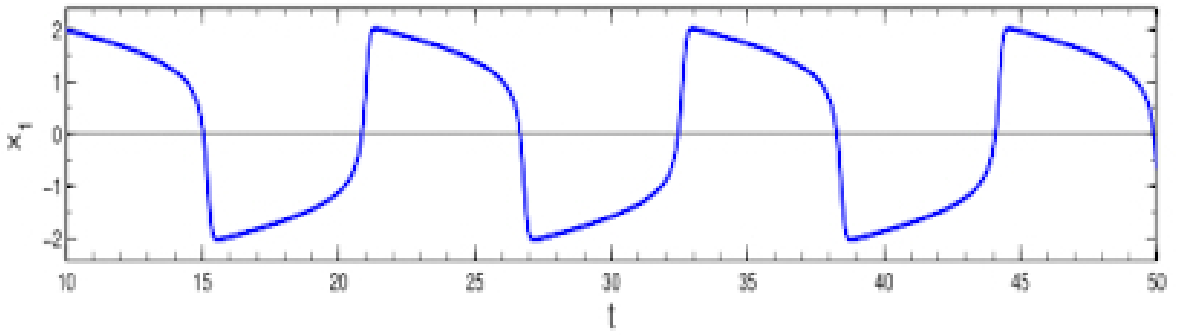
The system eventually stabilizes in an oscillation that equilibrates both effects.



van der Pol oscillation flow. The limit cycle is the red curve.

The system eventually stabilizes itself in a selfsustained oscillation that equilibrates both effects.

The oscillation is non-harmonic (relaxation oscillator) with 2 cycles.



We can transform the second order van der Pol equation to a system of 2 first order equations.

Let's make $y = \dot{x}$,

$$\begin{aligned} \dot{x} =& y \equiv f(x, y), \\ \dot{y} =& \mu(1 - x^2)y - x \equiv g(x, y). \end{aligned} \tag{38}$$

The Jacobian is

$$J = \begin{pmatrix} 0 & 1 \\ -1 - 2\mu xy & \mu(1 - x^2) \end{pmatrix} \tag{39}$$

It has only one equilibrium point $(0, 0)$ where J

$$J = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \tag{40}$$

The eigenvalues are

$$det \begin{pmatrix} -\lambda & 1 \\ -1 & \mu - \lambda \end{pmatrix} = \lambda^2 - \mu\lambda + 1 = 0 \quad (41)$$

and then

$$\lambda_{\pm} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2} = \frac{\mu}{2} \pm \sqrt{\left(\frac{\mu}{2}\right)^2 - 1}. \quad (42)$$

There are three bifurcations.

Starting with a very negative μ, the 2 eigenvalues are both real and negative.

In that case $(0, 0)$ is a stable node.

As μ grows there is a bifurcation point at $\mu = -2$ where the node becomes a stable spiral.

From there on the radicand becomes negative and the square root imaginary, with the eigenvalues becoming complex conjugates.

A second bifurcation appears when $\mu = 0$ with the real part going from negative to positive, and the spiral from stable to unstable.

For larger values of μ the unstable spiral becomes again an unstable node.

But the limit cycle around remains. Only the shapes of the trajectories near the origin change.

The van der Pol oscillator is a model describing a circuit with electronic valves in it.

It is related to the Rayleigh oscillator which described the oscillations of the clarinet's reed or the string on a violin.

The Rayleigh equation

$$\ddot{x} + kx - \mu\dot{x}(1 - \dot{x}^2) = 0, \qquad (43)$$

The non-linearity comes from a coefficient that changes the sign not with the amplitude but with the velocity ($\mu(1 - \dot{x}^2)$).

Linear analysis of a limit cycle

There are many different mechanisms for the creation of limit cycles.

One of them is the Hopf bifurcation.

$$\begin{aligned} \dot{r} &= \mu r(t) - r(t)^3, \\ \dot{\theta} &= \omega - b r(t)^2. \end{aligned} \tag{44}$$

where μ, ω and b are parameters.

The angular velocity $\dot{\theta}$ has a constant part ω. Looking at its functional shape with $r(t)$ it seems that it could oscillate around the origin.

But it also depends on the amplitude of the oscillation, a difference with harmonic oscillators.

From the radial equation we have $r^* = 0$ and $r = \sqrt{\mu}$.

The derivative df/dr is now $\mu - 3r^2$, and $r^* = 0$ could be unstable or stable depending on the sign of μ.

If $\mu < 0$ the origin is stable and the other equilibrium does not exist.

If $\mu > 0$ the origin is unstable and there is another stable equilibrium in $r^* = \sqrt{\mu}$. In the radial direction this is a pitchfork bifurcation.

In cartesian coordinates

$$\begin{aligned} \dot{x} &= [\mu - (x^2 + y^2)]x - [\omega + b(x^2 + y^2)]y, \\ \dot{y} &= [\mu - (x^2 + y^2)]y + [\omega + b(x^2 + y^2)]x. \end{aligned} \tag{45}$$

The Jacobian in the origin is

$$J = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \tag{46}$$

with eigenvalues $\lambda = \mu \pm i\omega$.

The oscillatory part of course is responsible for the oscillations.

The real part changes sign controlled by the sign of μ.

The eigenvalues cross the imaginary axis when μ goes from negative to positive.

This is a Hopf bifurcation.

In general the trajectory of the eigenvalues in the complex planes can change when the parameter responsible for the bifurcation changes.

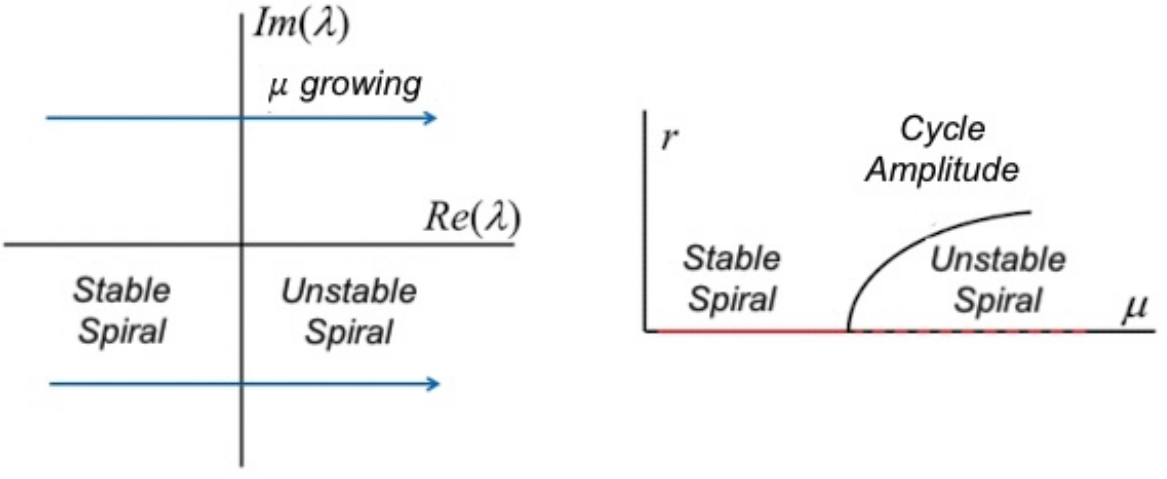
In the Hopf case there are just horizontal straight lines.

But the phenomenon is always the same: two complex conjugates eigenvalues cross the imaginary axis at the same time causing the instability of a stable spiral.

Near the bifurcation the amplitude is directly $\sqrt{\mu}$.

The frequency of oscillations, near the bifurcation is ω and the motion is harmonic.

As μ grows the cycle grows and motion will become less harmonic.



Hopf bifurcation.

Pendulum Phase space

The equation is

$$\ddot{\theta} + \frac{g}{l} \sin\theta = 0 \tag{47}$$

Let's do the change of variables $\tau = \omega t$, where $\omega = \sqrt{g/l}$

$$\ddot{\theta} + \sin\theta = 0 \tag{48}$$

which can be converted in a dynamical system of the first order:

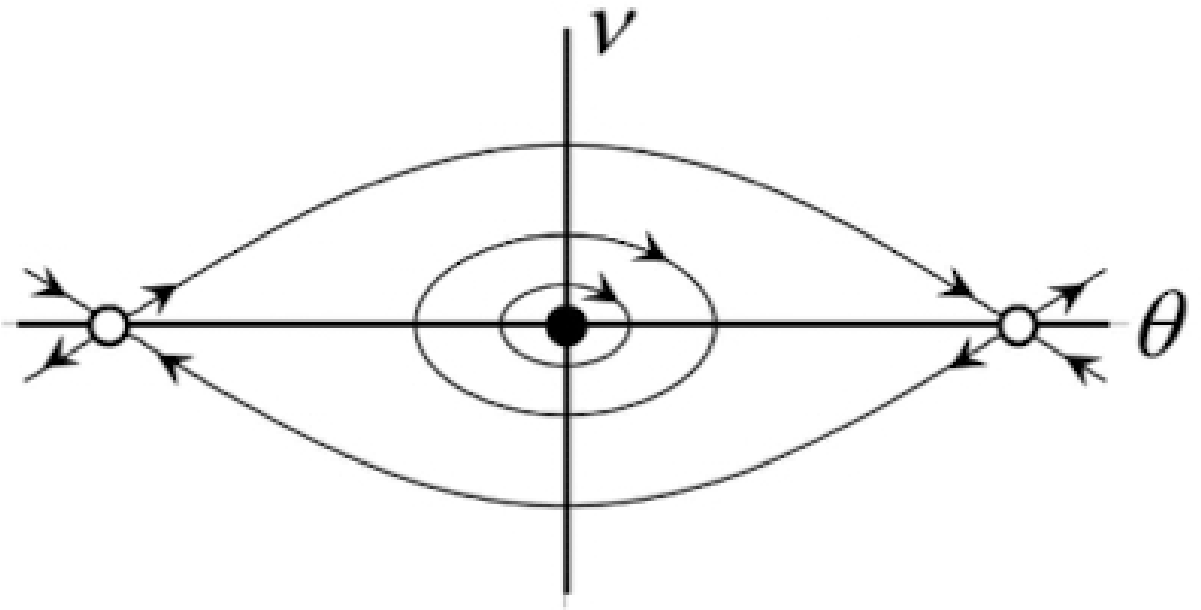
$$\begin{aligned} \dot{\theta} =& \nu, \\ \dot{\nu} =& -\sin\theta, \end{aligned} \tag{49}$$

ν is the non-dimensional angular velocity.

The fixed points are $(\theta^*, \nu^*) = (k\pi, 0)$, where k is an integer number.

It is easy to calculate the Jacobian of the linearized system at each point and physically we can concentrate in one cycle. We need to just look at $(0, 0)$ and $(\pi, 0)$, and find that the first is a center and the other one a saddle point.

The phase plot near the equilibrium points is like in the figure.



Phase space of linearized pendulum.

How can we complete the phase spot in the region far from equilibrium where the behavior is non linear?

Energy is conserved so we can plot the energy contours for different values of $E - \frac{1}{2}\nu^2 - \cos\theta$.

The lowest energy state corresponds to equilibrium at $(0, 0)$.

With more energy, near equilibrium, ellipses corresponding to the approximately harmonic system.

With more energy the ellipses deform approaching the stable and unstable manifolds around the saddle point.

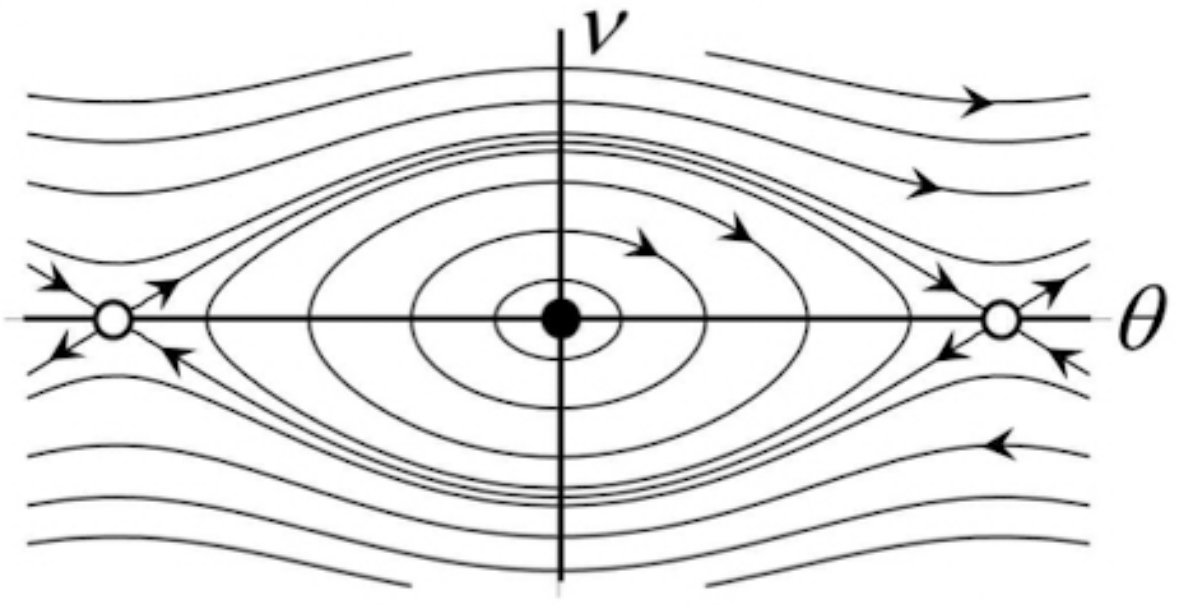
For a given value of energy unstable and stable manifolds interconnect.

This corresponds to the right amount of energy to reach unstable equilibrium from above (approaching through the stable manifold).

From there on a perturbation takes away the pendulum from equilibrium along the unstable manifold.

Even larger energies do not go through equilibrium points.

They correspond to a pendulum so energetic that would go around without returning points.

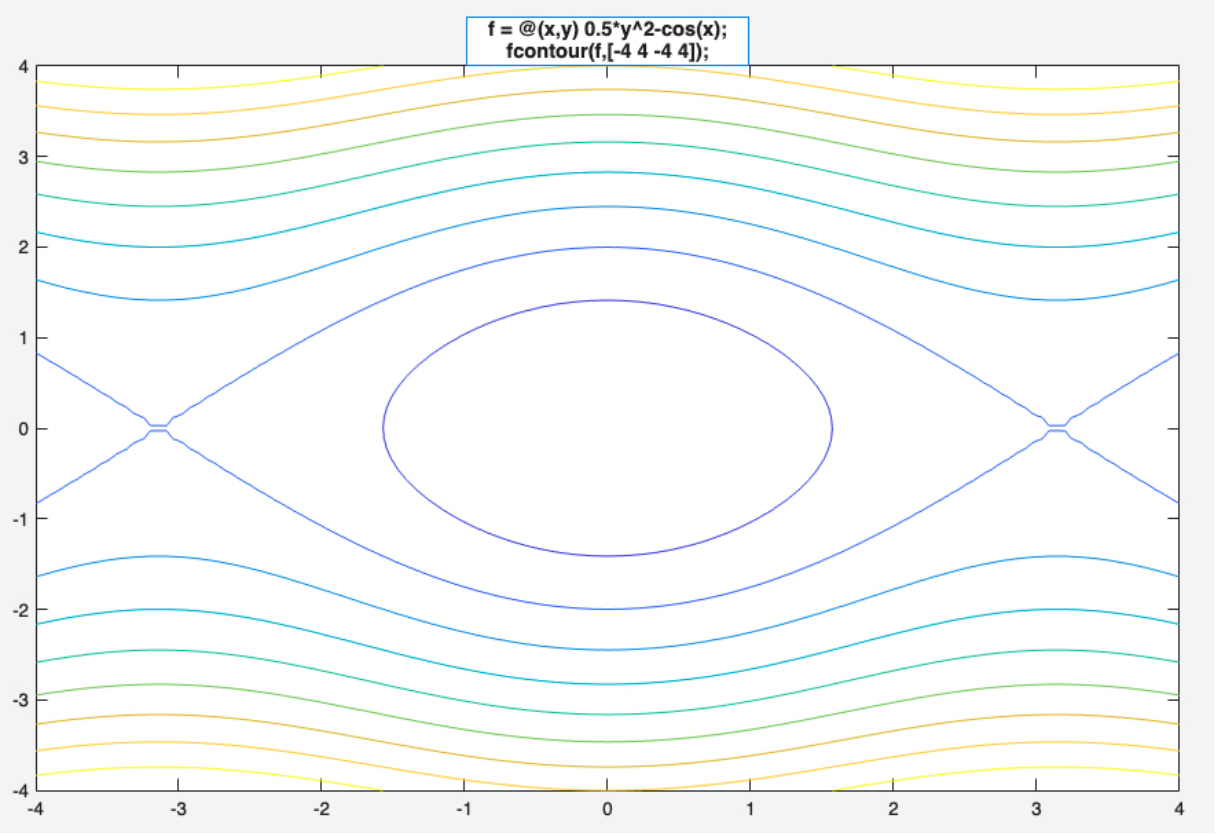


Phase space of a non-linear pendulum.

The maximum angular velocity corresponds to the pendulum going through $\theta = 0$ and the minimum when it goes through $\theta = \pi$.

The closed orbits are called **librations**, the trajectories that connect equilibrium points are called **heteroclinic curves**, and the open trajectories are called **rotations**.

Rotations are also periodic if we consider phase space periodic in the direction θ. We could fold it in a cylindric phase space.


f = @(x,y) 0.5*y^2-cos(x);
fcontour(f,[-4 4 -4 4]);
4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4

Chaos

In 3-D, a dynamical nonlinear autonomous system can evolve in a radically different way: **chaos**.

One branch of study is chaos in dissipative systems: hurricanes, tornadoes, turbulent flow, simple mechanical oscillators with friction, etc.

The other studies chaos in non-dissipative systems: hamiltonian chaos.

Chaos in dissipative systems was discovered in the second half of XX century: it requires dynamical system of at least dimension 3.

Hamiltonian system chaos was discovered by Poincare in the dynamics of 3 bodies interactions .

Non linear systems

The harmonic oscillator

$$m\ddot{x} = -kx, \tag{50}$$

is linear: no chaos.

The pendulum is nonlinear

$$ml^2\ddot{\phi} = -mgl \sin \phi. \tag{51}$$

as well as a planet around the Sun:

$$m\ddot{\vec{r}} = -GmM\frac{\widehat{r}}{r^2}. \tag{52}$$

But neither of them are chaotic. Nonlinearity is a necessary condition for chaos but not sufficient. Even a damped pendulum is not chaotic.

$$ml^2\ddot{\phi} = -mgl \sin \phi - \gamma l^2 \dot{\phi}, \tag{53}$$

It has a fixed point in $(0, 0)$ in phase space, which is a stable spiral and attractor of dynamics.

What about pushing it?

$$ml^2\ddot{\phi} = -mgl\sin\phi - \gamma l^2\dot{\phi} + lf(t). \qquad (54)$$

And this system, for some values of the parameters is chaotic.

What is special about it?

We can compare with linear systems where the principle of superposition holds.

In nonlinear systems this is not true. See exercise 6 HW.

The damped and forced pendulum

$$ml^2\ddot{\phi} = -mgl\sin\phi - \gamma l^2\dot{\phi} + lf(t). \qquad (55)$$

Let's assume that the driving external force is periodic with frequency ω and that it is harmonic

$$f(t) = f_0\cos(\omega t), \qquad (56)$$

the equation becomes

$$ml^2\ddot{\phi} + mgl\sin\phi - \gamma l^2\dot{\phi} = lf_0\cos(\omega t), \qquad (57)$$

Dividing by ml^2 and defining an absorption $2\beta = \gamma/m$

$$\ddot{\phi} + 2\beta\dot{\phi} + \frac{g}{l}\sin\phi = \frac{f_0}{ml}\cos(\omega t), \qquad (58)$$

The natural frequency of the pendulum is $\omega_0 = g/l$.

Notice that

$$\frac{f_0}{ml} = \frac{f_0}{mg}\frac{g}{l} = \frac{f_0}{mg}\omega_0^2 \equiv \alpha\omega_0^2 \qquad (59)$$

Where α is the ratio of the external force over the weight (forcing intensity). Then:

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin\phi = \alpha\omega_0^2 \cos(\omega t), \qquad (60)$$

If we write it as an autonomous dynamical system

(no explicit dependence of time) we get

$$\begin{aligned}\dot{\phi} =& v \\ \dot{v} =& -2\beta v - \omega_0^2 \sin\phi + \alpha\omega_0^2 \cos(\theta) \\ \dot{\theta} =& \omega,\end{aligned} \tag{61}$$

and all the right hand side is an $F(\phi, v, \theta)$ independent of t.

What can we guess before any calculation?

If $\alpha < 1$ the external force is lower than the weight of the system and we should expect a small amplitude. f_0 is not enough to have the pendulum reach $90°$. An almost linear regime.

But if $\alpha > 1$ the external force is larger than the weight of the system and we can expect the system to turn around.

A domain of large amplitudes and strong non-linearity.

Review of the linear oscillator

Let's make the initial amplitude small with a low initial velocity. If $\alpha \ll 1$ we expect oscillations to remain small:

$$\sin\phi \approx \phi, \tag{62}$$

With the equation becomes

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\phi = \alpha\omega_0^2 \cos(\omega t), \qquad (63)$$

The solution is easier to be found if we look at this equation

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\phi = \alpha\omega_0^2 e^{i\omega t}, \qquad (64)$$

if we look for particular solution of the inhomogeneous equation in the form

$$\phi = Be^{i\omega t}, \qquad (65)$$

we get for B

$$B = \alpha(\omega_0^2 - \omega^2 + 2i\beta\omega), \qquad (66)$$

We write now $B = be^{i\delta}$, and get

$$b = \alpha\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}, \quad \tan\delta = \frac{2\beta\omega}{\omega^2 - \omega_0^2} \tag{67}$$

Taking the real part of B we find the particular integral of (63) and adding to the general solution of the homogeneous equation we get

$$\phi(t) = ae^{-\beta t}\cos(\Omega t + \alpha) + b\cos(\omega t + \delta). \tag{68}$$

where $\Omega = \sqrt{\omega_0^2 - \beta^2}$ and α an arbitrary initial constant. But we see that $ae^{-\beta t}\cos(\Omega t + \alpha)$ is

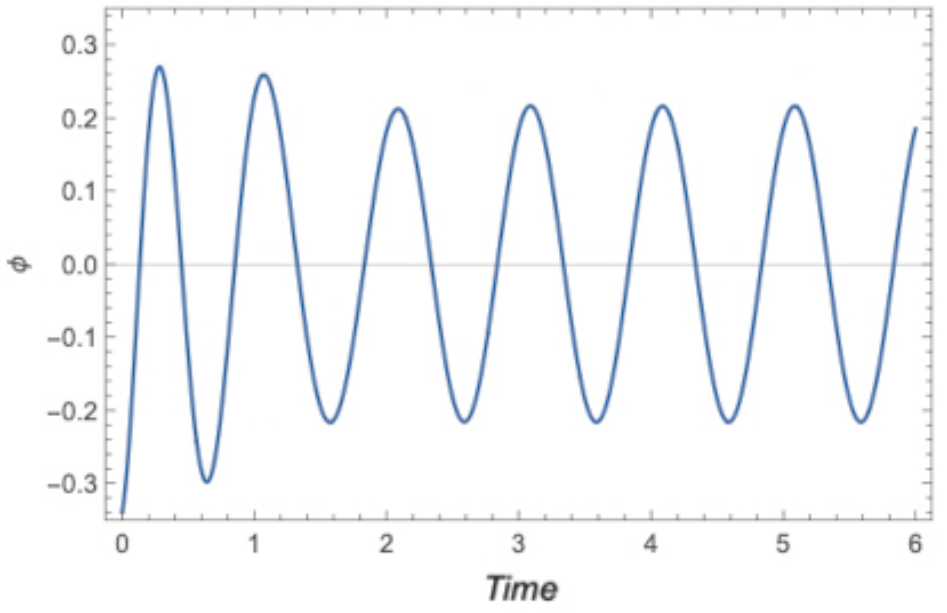
transient regime that would fade away fast and motion will converge to an attractor that oscillates with the frequency of the external force:

$$\phi(t) = b\cos(\omega t + \delta). \tag{69}$$

In summary:

1) There is an attractor.

2) The frequency of the attractor is ω.



The x axis is in units of cycles

Quasi linear oscillations

We increase the external force, but now we cannot claim $\sin\phi \sim \phi$.

If we go the next order in a Taylor series:

$$\sin\phi \approx \phi - \frac{1}{6}\phi^3. \tag{70}$$

So to the next order the equation becomes

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\left(\phi - \frac{1}{6}\phi^3\right) = \alpha\omega_0^2\cos(\omega t), \tag{71}$$

If the amplitude of the oscillation is not too large, the term ϕ^3 shouldn't be too large either.

We expect then a situation similar to the previous equation:

$$\phi(t) \approx b\cos(\omega t + \delta). \tag{72}$$

We can put this(72) in (71) the term ϕ^3 will generate

$$\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x). \tag{73}$$

This makes the equation look like

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\phi - \frac{1}{6\times 4}\Big(3\cos(\omega t - \delta) + \cos 3(\omega t - \delta)\Big) = \alpha\omega_0^2\cos(\omega t), \tag{74}$$

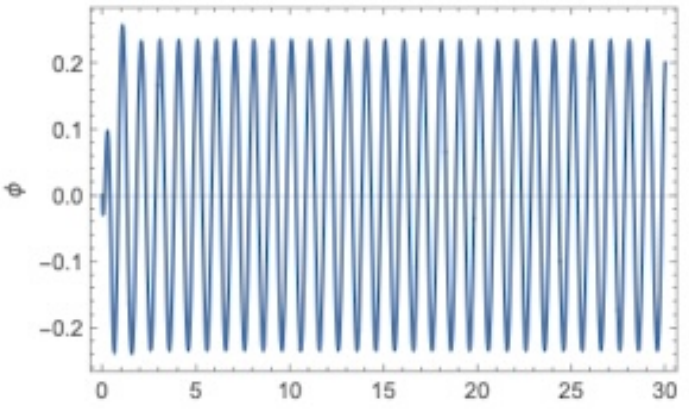
We see in the left hand side of the equation a term with a third harmonic which does not appear in the right one.

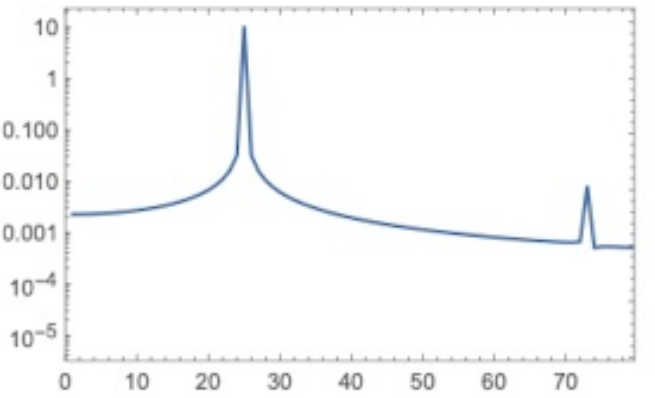
This means that we need either ϕ, or $\dot{\phi}$, or $\ddot{\phi}$ (or all of them at once) to have a third harmonic in $\cos \omega t$.

We need something like

$$\phi \approx A \cos(\omega t - \delta) + B \cos 3(\omega t - \delta). \qquad (75)$$

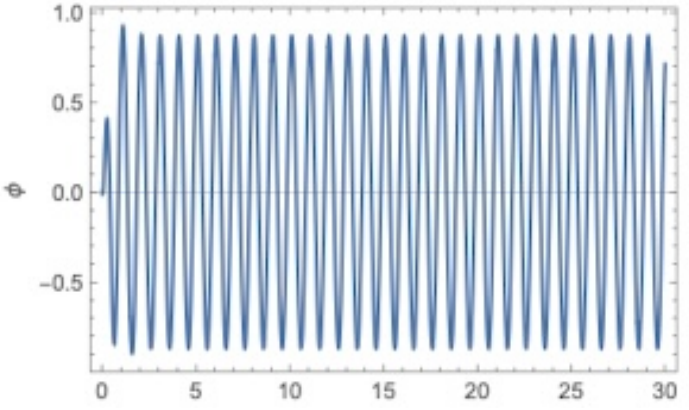
with $B \ll A$. So as α increases the amplitude will as well and a third harmonic will appear with frequency 3ω.

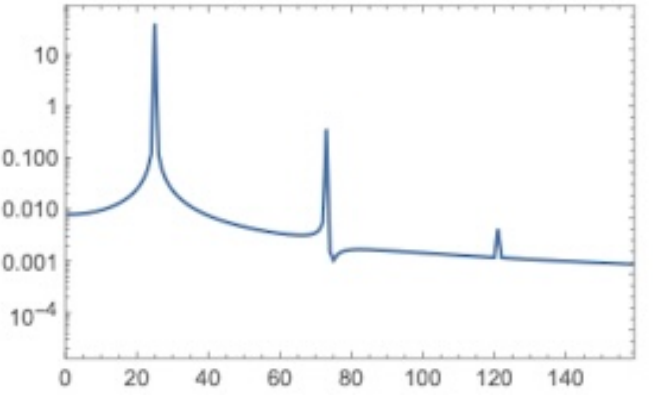




Almost linear regime with the Fourier transform on the right ($3\omega = 72$).

We can now repeat the argument



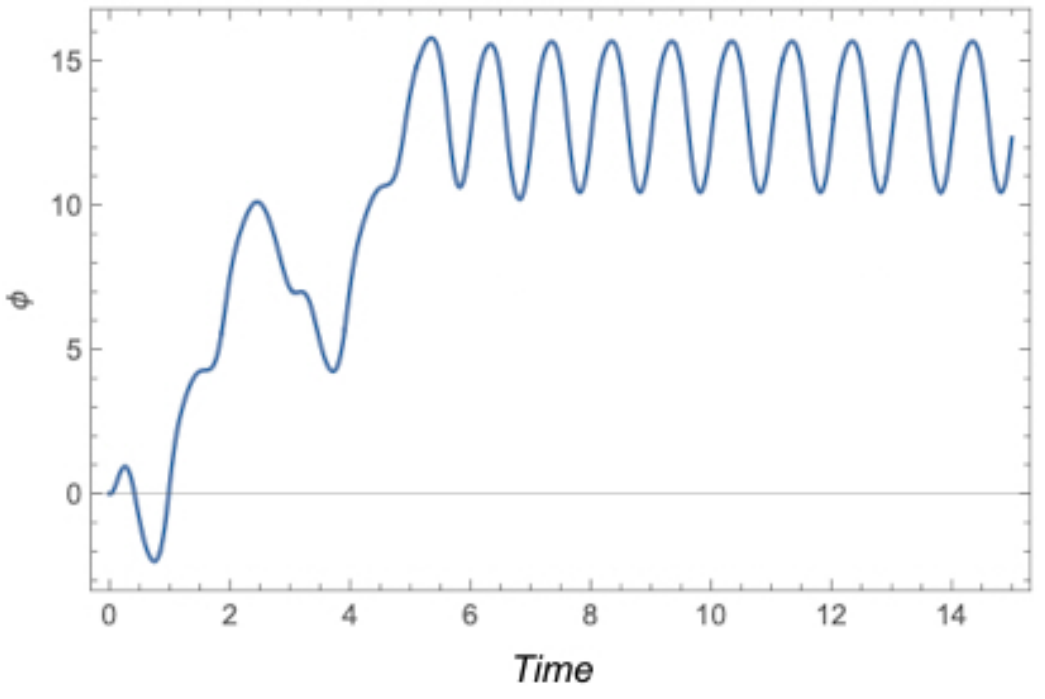


The Fourier transform shows a fifth harmonic ($5\omega = 120$). $\alpha = 0.5$.

The conclusion is:

As the intensity of the external force increases, the solution becomes less harmonic, incorporating higher harmonics from the external frequency.

The appearance of higher harmonics is typical of non-linear systems. This higher harmonics contain frequencies that are multiples of the forced one. this means that when the fundamental completes a circle, the nth harmonic completes n-cycles. So the solution continues to be periodic of period $\tau = 2\pi/\omega$.



Right after superseding the linear regime $\alpha = 1.06$.

The path to chaos

If we keep increasing α (remember $\alpha = f_0/mg$), $\alpha > 1$ where the external force exceeds the pendulum weight.

Let's take $\alpha =$ as the limit. As we saw in the previous plot, the transient that before was short, now is longer. Additionally the pendulum has spun several times around.

The attractor still stays centered around $2k\pi$, which is the position that coincides with the pendulum hanging from the inferior point. It is impossible to be certain if motion continues to be periodic with period

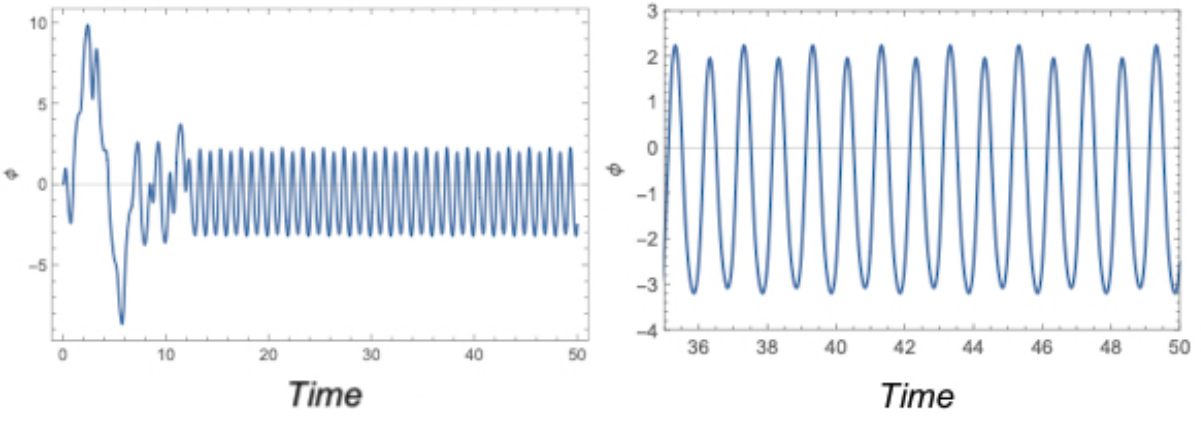
$2\pi/\omega$. It could be checked reviewing $\phi(t)$. (it would take going up to $t \approx 45$ to see repeating values).

Double period

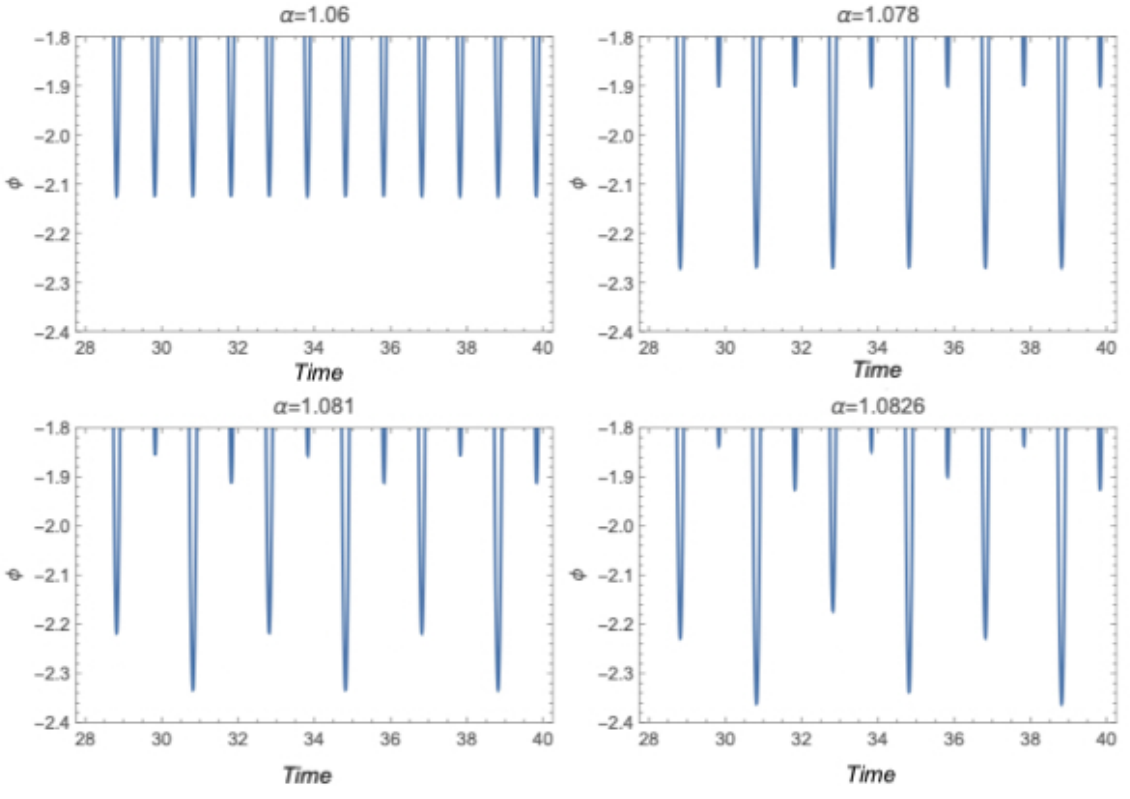
If we take α to a higher value: 1.073. We can see a long transient and new attractor that at first glance looks periodic.

But a closer look shows that the values repeat themselves every two cycles of the forced frequency.

The period is twice as large as it was before. Which means frequency wise it is a subharmonic of ω ($\omega/2$).



Double period $\alpha = 1.073$. $\omega/2$ visible.



Periods 2, 4 and 8.

Bifurcation cascade

Increasing α further, a sequence o bifurcations appears.

A period-doubling bifurcation occurs when a slight change in a system's parameters causes a new periodic trajectory to emerge from an existing periodic trajectory—the new one having double the period of the original.

With the doubled period, it takes twice as long for the numerical values visited by the system to repeat themselves.

A period-halving bifurcation occurs when a system switches to a new behavior with half the period of the original system.

A period-doubling cascade is an infinite sequence of period-doubling bifurcations.

Such cascades are one route by which dynamical systems can develop chaos.

In hydrodynamics, they are one of the possible routes to turbulence.

This phenomenon is surprising, but what is truly extraordinary is the manner in which the cascades are always the same for a very ample range of systems, independently of the details.

n	period	α_n	$\Delta\alpha$
1	$1 \to 2$	1.0663	
2	$2 \to 4$	1.0793	0.0130
3	$4 \to 8$	1.0821	0.0028
2	$8 \to 16$	1.0827	0.0006

This property is called universality. The table shows the first bifurcations. Each one is $\approx 1/5$ of the previous value.

In the 1970s Mitchell Feigenbaum showed that in all these systems the bifurcation points satisfy a geometric relationship

$$(\alpha_{n+1} - \alpha_n) \approx \frac{1}{\delta}(\alpha_n - \alpha_{n-1}), \qquad (76)$$

where $\delta = 4.6692016...$ is the constant δ of Feigenbaum (there is another Feigenbaum constant).

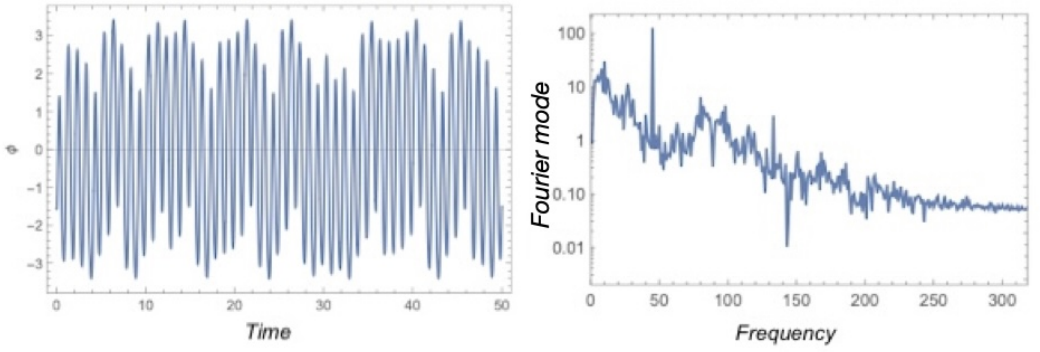
The $\approx$ sign is due to the fact that the relationship holds for $n \to \infty$. In that limit, all α_n converge. In the case of the Damped Forced Pendulum:

$$\alpha_n \xrightarrow{n\to\infty} \alpha_c = 1.0829... \qquad (77)$$

What happens after α_c is reached?

The solution stops being periodic and becomes chaotic.

The mechanism of bifurcation cascade is one of the possible routes to chaos (but not the only one).

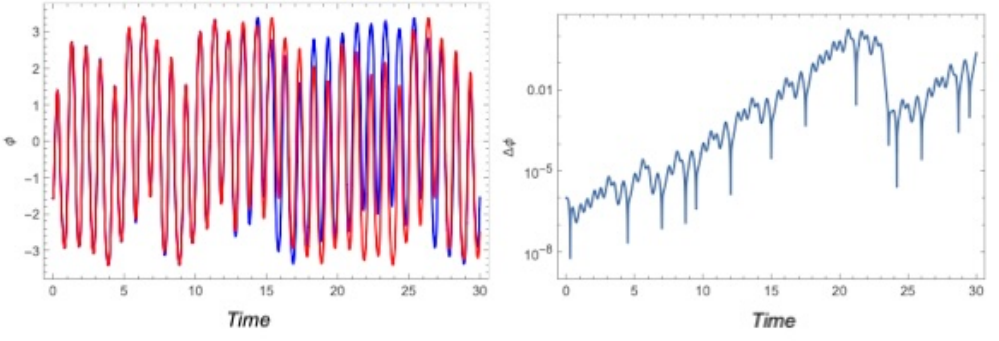


Chaotic regimen with $\alpha = 1.105$.

Chaos

The plot above shows a highly irregular non-periodic temporal behavior more clearly evidenced in the Fourier transform plot.

The other crucial characteristic essential to these systems is the sensitivity to initial conditions.



Above 2 solutions are plotted with $\alpha = 1.105 > 1.0829$ which differ initially in $\Delta\phi = \epsilon = 0.01$. As time passes their spectrum content differ greatly.

We can define

$$\Delta\phi(t) = \phi_2(t) - \phi_1(t), \tag{78}$$

where ϕ_1 and ϕ_2 are the trajectories corresponding to two closed initial conditions. For a linear oscillator $\Delta\phi \to 0$.

The general solution is

$$\begin{aligned} \phi(t) &= A\cos(\omega t - \delta) + C_1 e^{r_1 t} + C_2 e^{r_2 t}, \\ \Rightarrow \Delta\phi &= B_1 e^{r_1 t} + B_2 e^{r_2 t}, \end{aligned} \tag{79}$$

B_1 and B_2 depend on the initial conditions.

If we plot $\log|\Delta\phi(t)|$, we will see that it comes down.

It is decaying exponentially... While α is not too big, we can see that this is still the case.

With $\alpha = 1.07$ (which corresponds to a period 2) we can go to $t \approx 40$ and there is still decaying, although slowly.

But if we go over $\alpha_c = 1.0829..$ the behavior changes radically: $\Delta\phi$ grows.

In the Figure from the previous slides we can see $\alpha_c = 1.105$, and $\epsilon = 10^{-6}$: the difference between the two solutions grows exponentially. The difference can not grow larger than π, so in the end we see saturation.

This sensitivity to the initial conditions makes impossible to make a prediction as to the evolution of the system.

Mathematically, we need to observe at each point of the trajectory how the orbits diverge and look at

the behavior of:

$$\Delta\phi \sim e^{\lambda t}. \tag{80}$$

λ is the **Lyapunov exponent**.

$\lambda < 0$ Nearby trajectories converge toward a stable fixed point or a periodic limit cycle. The system is stable and predictable.

$\lambda = 0$ The system exhibits marginal stability, typically associated with a neutral state, a conserved quantity, or a continuous family of periodic orbits (limit cycles).

$\lambda > 0$ Nearby trajectories separate exponentially over time. This sensitive dependence on initial conditions is the hallmark of chaos.

The Lyapunov Spectrum

In an n-dimensional dynamical system, there are n distinct Lyapunov exponents (forming the Lyapunov spectrum), corresponding to the growth or contraction rates along different principal directions of an infinitesimal perturbation sphere.

The largest (or maximal) Lyapunov exponent (λ_1) is the most critical; i$\lambda_1 > 0$, the system is generally

considered chaotic because exponential stretching in that single direction eventually dominates the long-term behavior.

For a bounded chaotic attractor, the sum of all Lyapunov exponents must be negative, ensuring that phase space does not expand infinite.

Predictability Horizon:

The magnitude of a positive Lyapunov exponent sets a strict limit on how far into the future a system can be reliably predicted.

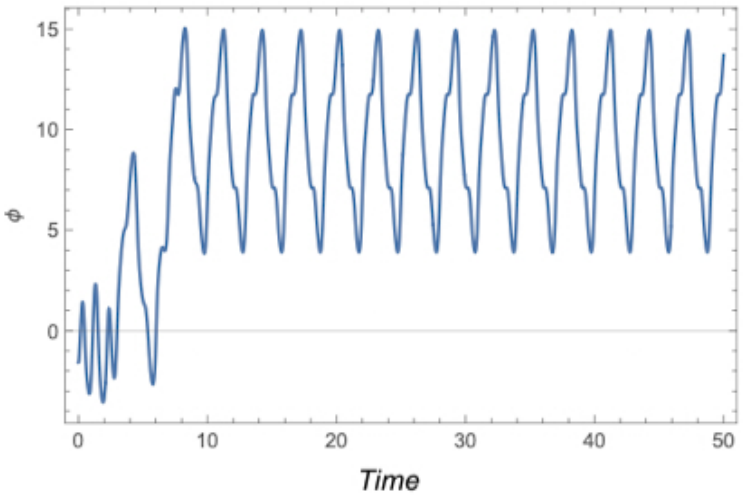
For example, weather systems have positive Lyapunov exponents, which define why long-term forecasting breaks down past a certain window.

Regularity windows

Going back to our system, the damped forced oscillator, if we keep increasing α, we can get surprises.

i.e. $\alpha = 1.13$ we find a window of period 3.

How can we get from a period 1 to a period 3?



A theorem by Yorke and Li (1975) shows that period 3 implies chaos.

For $\alpha = 1.503$ the solution is again chaotic, but in this case the pendulum goes around many times.

Bifurcation diagrams

A bifurcation diagram shows all these different behaviors in one plot.

The vertical axis represents the trajectory and the horizontal one the control parameter (like α).

Remember slides 18 and 19 where we looked at $\dot{x} = rx - x^3$ and plotting x vs r we observed the pitchfork diagram.

The situation is a bit different though.

We don't have equilibrium points now.

We are interested in seeing if the solution is periodic or no, and with what period, so can look at:

$$\phi(t_0),\ \phi(t_0+1),\ \phi(t_0+2),... \tag{81}$$

for some t_0 large enough so that the transient had already occurred.

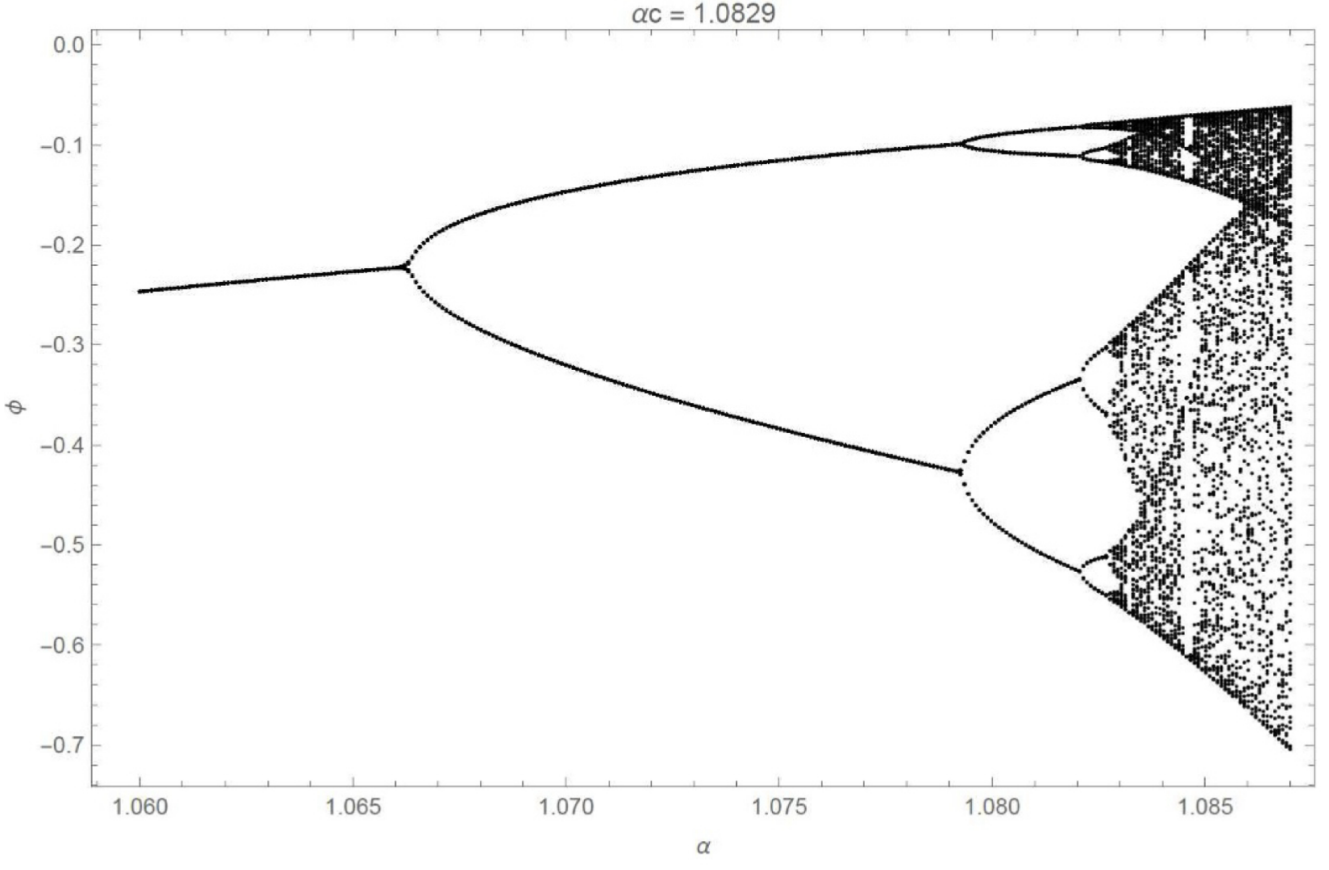
If the trajectory is periodic of period n, there will be n different values.

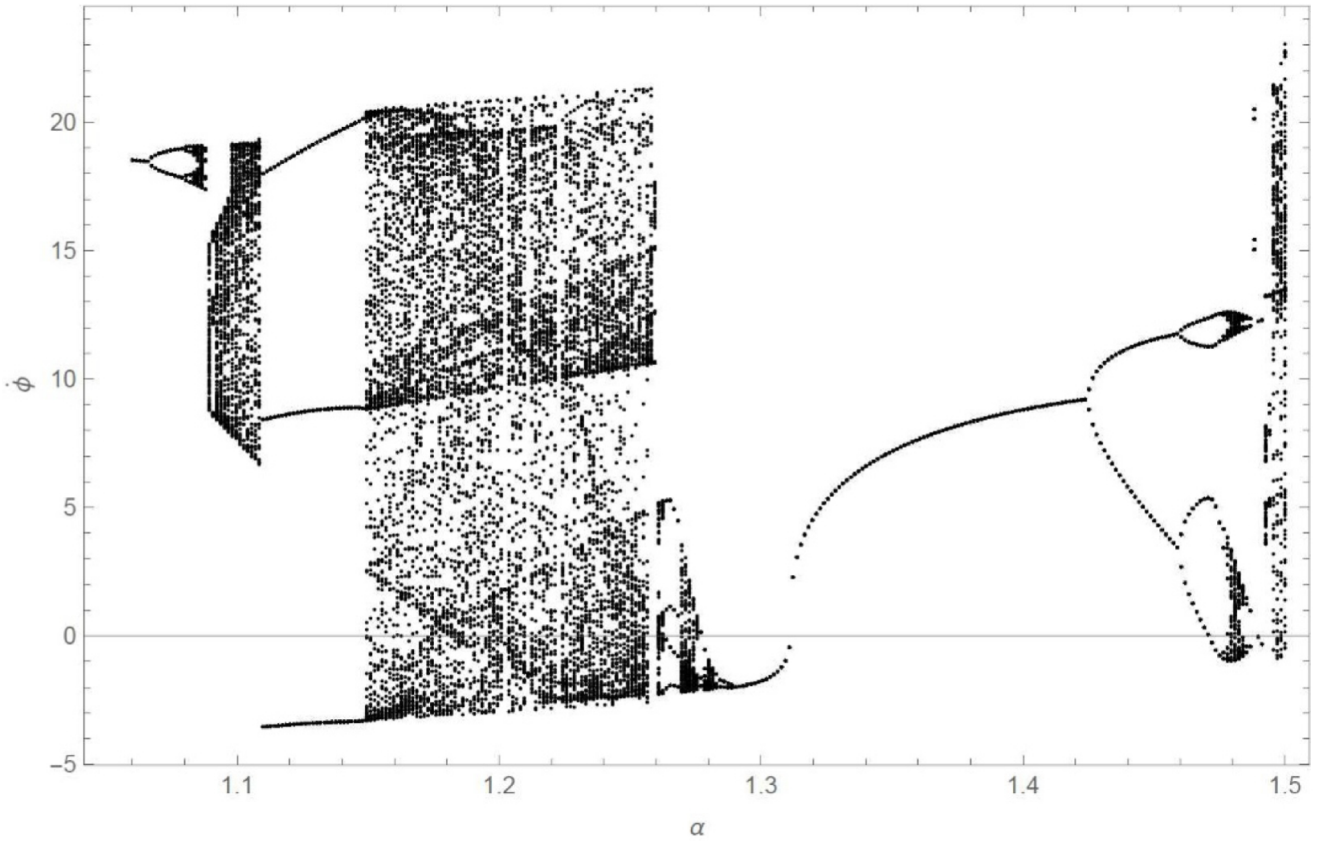
If the trajectory is chaotic, we observe a completely disordered scattering of values, without any discerning pattern.

The next slide shows the result with different values of α up to 1.087.

When α is too large, the force is too large and the pendulum spins around, with growing limitless.

We could do a plotting of $\dot{\phi}$ instead which will remove ambiguity. This is the slide after next.


αc = 1.0829
0.0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−0.7
ϕ
1.060
1.065
1.070
1.075
1.080
1.085
α


20
15
10
5
0
−5
$\dot{\phi}$
1.1
1.2
1.3
1.4
1.5
α

Condition necessary for Chaotic behavior

The system

$$\dot{x}(t) = x(t), \tag{82}$$

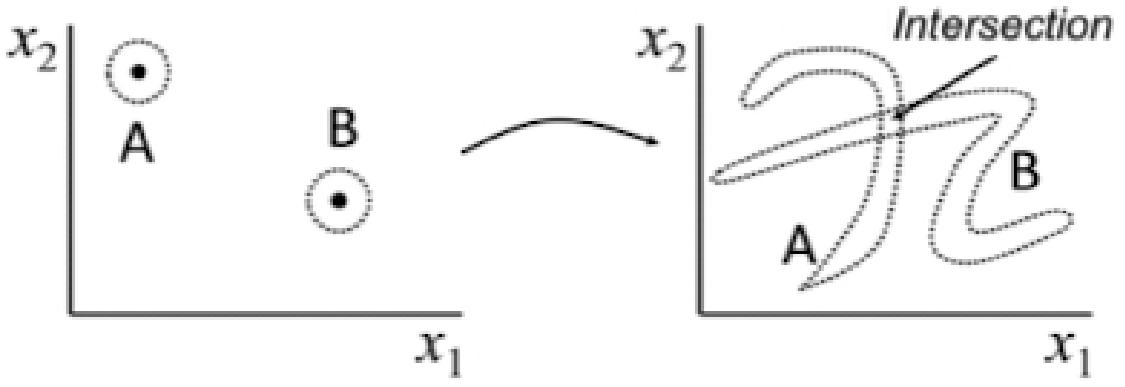
has sensitivity to initial conditions: trajectories with different initial conditions differ exponentially.

But the system is not chaotic.

There other two conditions that are necessary for dynamical systems to evolve chaotically.

Mixing:

If we look at two disjoint open sets of initial conditions, the evolution make their intersection non-null. Initial conditions get mixed.



Dense trajectories:

A crucial element of chaos in dynamical systems, is to have a dense trajectory.

This means that the system, through its trajectory, covers an orbit that comes arbitrarily close to every point in the phase space.

In the literature this is defined as Topological Transitivity.

A system is topologically transitive if, for any two open sets in the phase space, a trajectory eventually travels from one set into the other.

In summary, for a system to be chaotic, it has to have

1. Sensitive dependence on initial conditions (the butterfly effect, characterized by a positive Lyapunov exponent).

2. Exhibit topological transitivity (which is related to the presence of a dense trajectory).

3. Dense periodic orbits (meaning periodic trajectories are scattered everywhere densely throughout the system's attractor).

Chaos in Hamiltonian Systems

One of the few things we can say about Hamiltonian systems in 2-D is that they preserve the volume in phase space.

Otherwise they are quite complex. For example a hamiltonian:

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2m} + U(q_i), \tag{83}$$

that conserves the systems energy satisfies also

$$H(q_i, p_i) = E. \tag{84}$$

So the trajectories in the 2-d dimensional phase space are restricted to a $2n-1$ submanifold of it.

Let's look at

$$H = \frac{1}{2}(p_x^2 + p_y^2) + U(x,y). \tag{85}$$

Once we know the initial positions and momenta, the trajectory is determined by

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x}, & \dot{p_x} &= -\frac{\partial H}{\partial x}, \\ \dot{y} &= \frac{\partial H}{\partial p_y}, & \dot{p_y} &= -\frac{\partial H}{\partial y}, \end{aligned} \tag{86}$$

The phase space is $4D$, but conservation of energy constrains the four variables and restricts the trajectories to a 3-D submanifold:

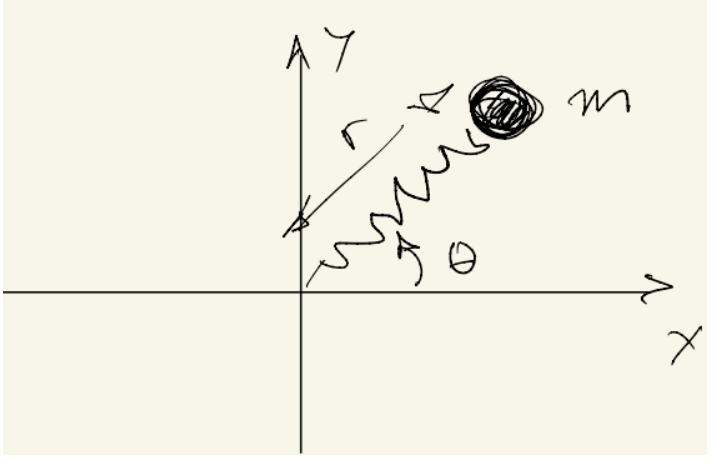
$$H(x, y, p_x, p_y) = E. \tag{87}$$

We can assume that if the energy is small enough the trajectories will be limited to certain region of phase space.

The orbits will be confined.

A special class of hamiltonian systems are integrable systems.

Let's look at an example:



$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}kr^2, \qquad (88)$$

The system is time invariant and the energy is conserved

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}kr^2, \tag{89}$$

θ being cyclic gives conservation of momentum

$$M = mr^2\dot{\theta}, \tag{90}$$

The hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{1}{2}kr^2 = E \tag{91}$$

With this example in mind let's define some stuff.

Involution

Two functions F_1 and F_2 are in **involution** if their Poisson bracket is 0.

$$[F_1, F_2] = 0 \tag{92}$$

Integrable systems

If a system with n degrees of freedom has n conserved quantities that are in involution, then the system is integrable.

If the conserved quantities of the system are denoted by

$$C_i = C_i(q, p) \tag{93}$$

With $i = 1...n$ each being a function of all the $2n$ phase space coordinates, then the system is integrable if

$$[C_i, C_j] = 0 \qquad \forall i, j. \tag{94}$$

Going back to our example we see that:

H and p_θ being our two quantities that are conserved

$$[H, p_\theta] = \frac{\partial H}{\partial r}\frac{\partial p_\theta}{\partial p_r} - \frac{\partial H}{\partial p_r}\frac{\partial p_\theta}{\partial r} + \frac{\partial H}{\partial \theta}\frac{\partial p_\theta}{\partial p_\theta} - \frac{\partial H}{\partial p_\theta}\frac{\partial p_\theta}{\partial \theta} = 0 \tag{95}$$

Also

$$[H, H] = 0 \qquad [p_\theta, p_\theta] = 0. \tag{96}$$

Then the system in our example is then integrable.

The Hamilton equations are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}. \tag{97}$$

$$\dot{p_\theta} = -\frac{\partial H}{\partial \theta} = 0 \qquad p_\theta = constant. \tag{98}$$

From (97) we get

$$\int d\theta = \int \frac{p_\theta}{mr^2} dt. \tag{99}$$

and

$$\dot{r} = \frac{\partial H}{\partial p_r} \qquad \dot{p_r} = -\frac{\partial H}{\partial r}. \tag{100}$$

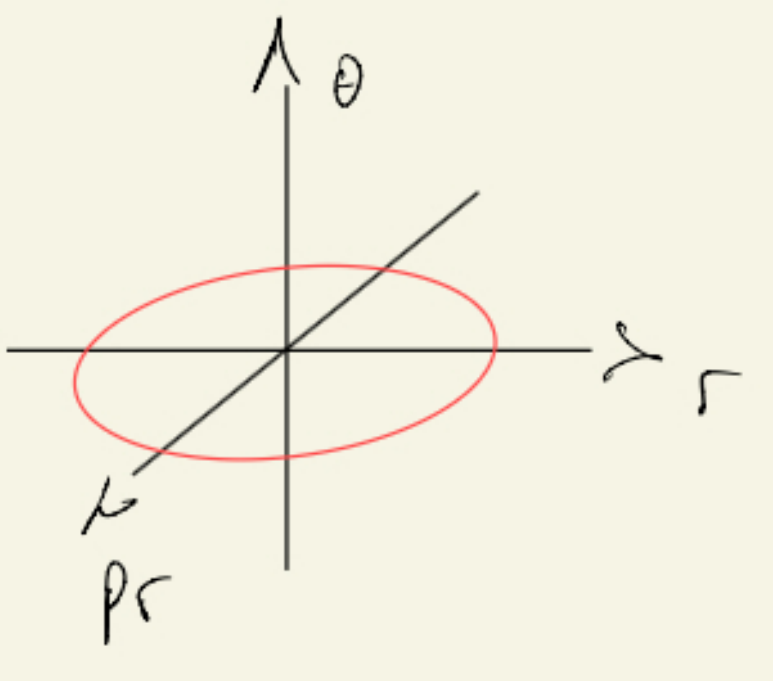
which gives

$$\dot{r} = \frac{p_r}{m} \qquad \dot{p_r} = -\frac{p_\theta^2}{mr^3} - kr. \tag{101}$$

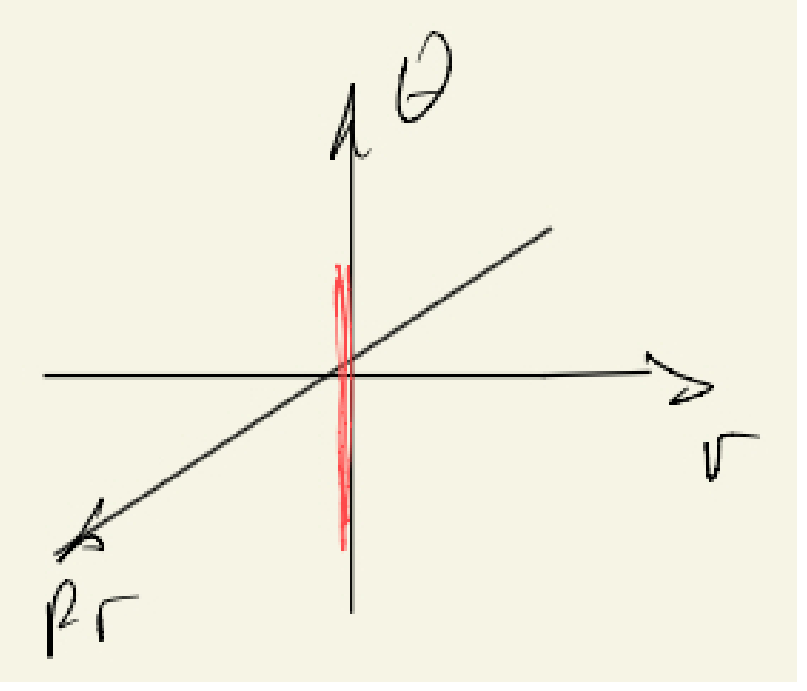
So

$$t = \int \frac{mrdr}{\sqrt{2Emr^2 - p_\theta^2 - kmr^4}}. \tag{102}$$

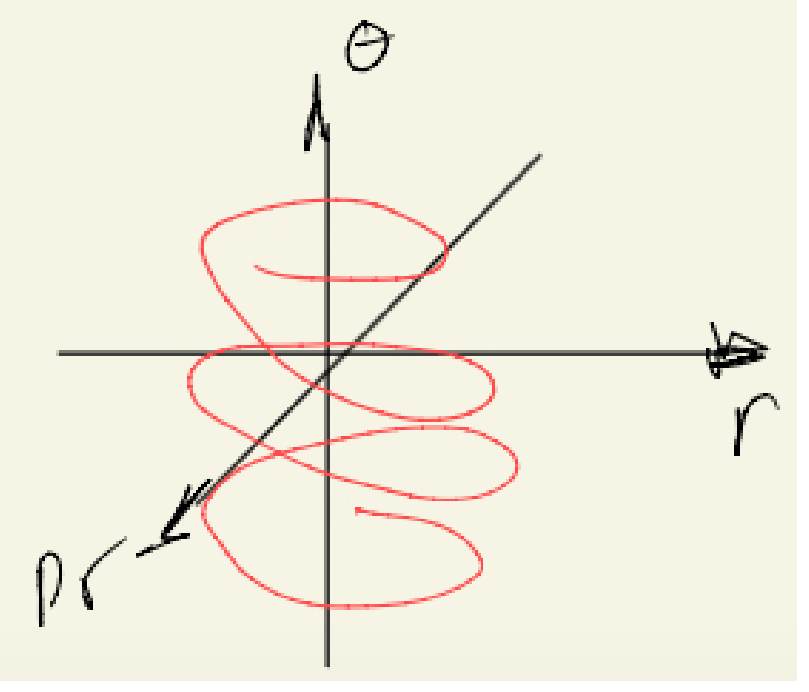
$$p_r = \int \left(\frac{p_\theta^2}{mr^3} - kr\right) dt. \tag{103}$$



a) Trajectory with $p_\theta = 0$, $\theta = constant$.

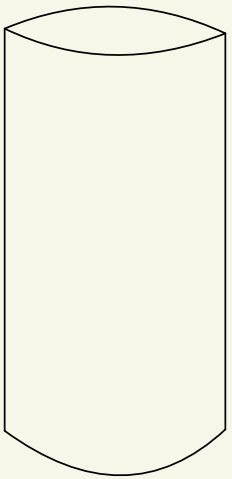


b) Trajectory with $\dot{\theta} = \sqrt{k/m}$, $\ddot{\theta} = 0 \rightarrow p_r = 0$ ($r = ct$).

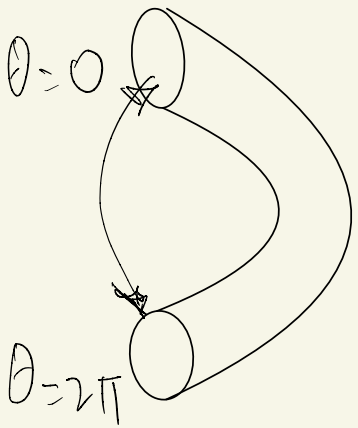


c) A generic trajectory.

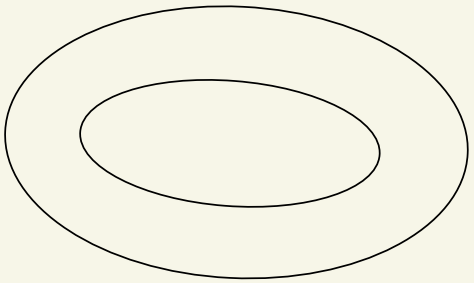
In general the trajectory takes place on a cylinder.



Cylinder



Identify 0 and 2 π

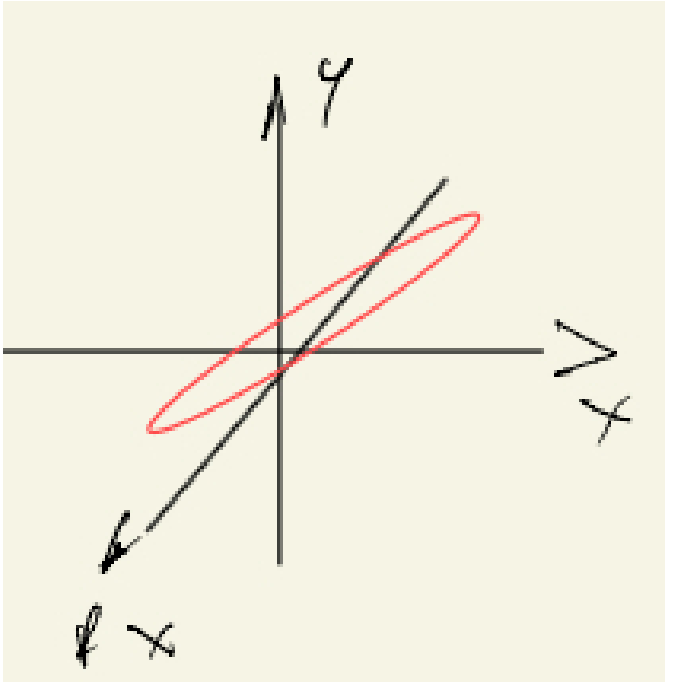


Torus

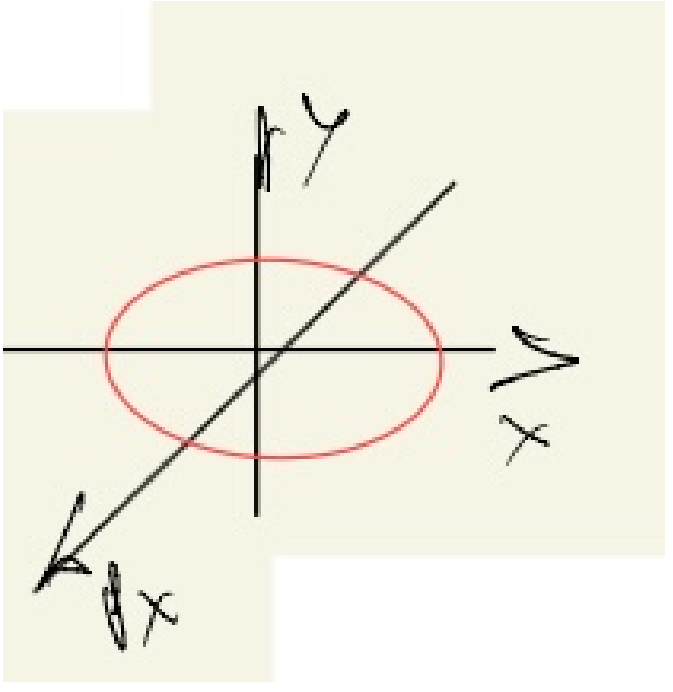
If we look at the same problem in Cartesian coordinates

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}kx^2 + \frac{1}{2}ky^2 \qquad (104)$$

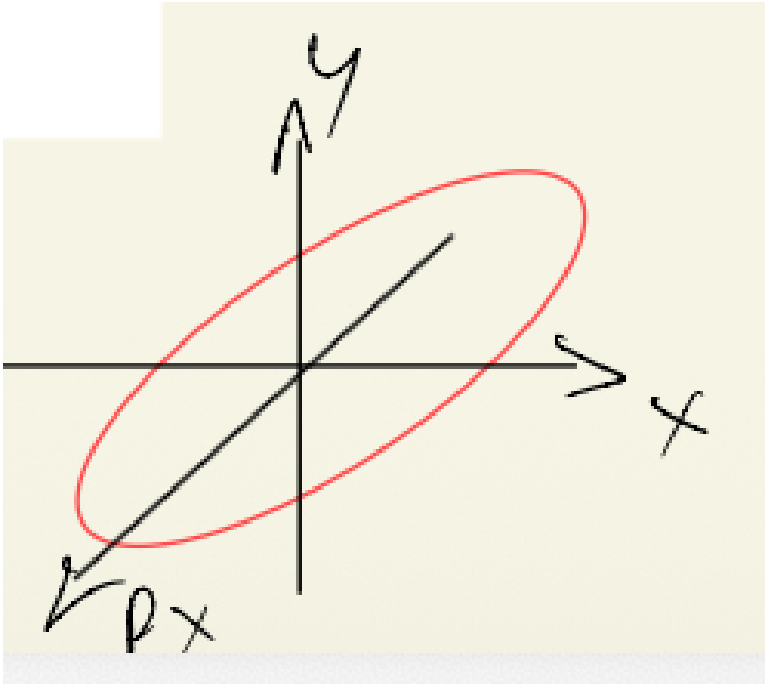
The problem decouples in two harmonic oscillators



$p_\theta = 0$, and motion is a line in the $x - y$ plane.



$\dot{\theta} = \sqrt{k/m}$ and. motion is in a circle.



The generic case.

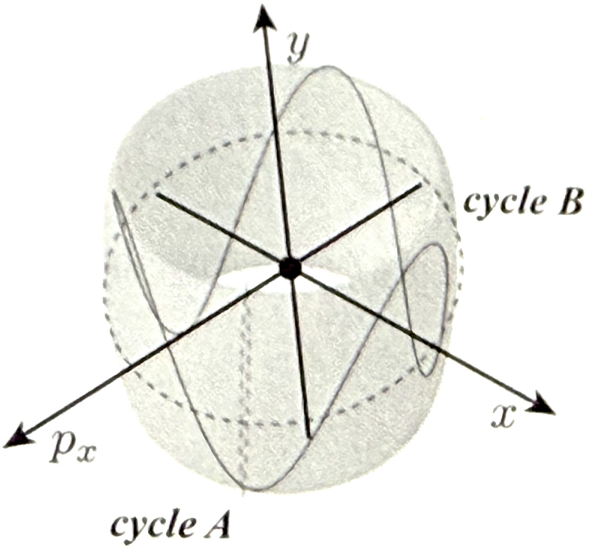
We can jump now into a double harmonic oscillator but this time with two different frequencies.

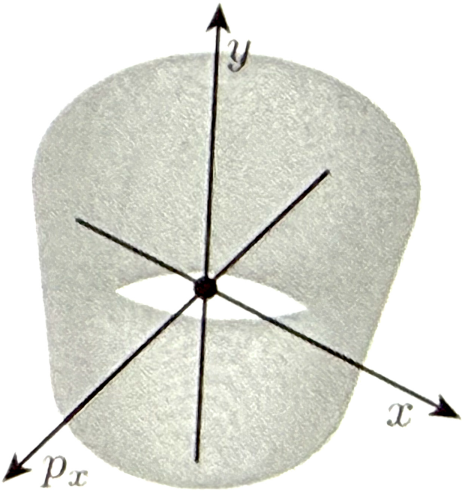
$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega_1^2 x^2 + \frac{1}{2}m\omega_2^2 y^2 \quad (105)$$

where $\omega_1 \neq \omega_2$.

The energy of this system is conserved.

We can define $\nu = \frac{\omega_1}{\omega_2}$. This is called the winding number. Typically $\omega_1 < \omega_2$, so $0 < \nu < 1$.





The left plot corresponds to $\nu = 1/3$ and the right one to an irrational value.

Integrable systems in 2 dimensional phase space follow this pattern.

The intersection of the N-subspaces defined by $C_i = C_i(q, p)$ is a torus in a $2N - N = N$ dimensional space. It is called an N torus.

It has N non contractible cycles and can be identified as an N dimensional hypercube identifying opposite sides.

The system is then characterized by N angular frequencies, one for each non-contractible cycle.

Depending on the relationship between these frequencies the trajectories may close or densely cover the torus.

Summarizing: Integrable systems are those that in addition to the conservation of energy have additional conserved quantities.

These additional constrains between momenta and coordinates restricts further the allowed trajectories which move in a 2-dimensional manifold.

This movement of integrable systems is called regular.

If there are more degrees of freedom that constants of motion, the trajectories can become chaotic.

There are two types of integrable systems: separable systems and central potentials.

In the case of separable systems, the potential is the sum of two independent function of the coordinates, ie.

$$U(x, y) = U_x(x) + U_y(y), \qquad (106)$$

Then the hamiltonian is the sum of two hamiltonians with one degree of freedom each

$$H = H_x + H_y; \qquad H_{x,y} = \frac{1}{2} p_{x,y}^2 + U_{x,y}, \tag{107}$$

Then, the motion in each coordinate becomes decoupled from the other and each hamiltonian is conserved separately.

In the case of central forces

$$U(x,y) = U(r); \qquad r = (x^2 + y^2)^{1/2}, \tag{108}$$

and the angular momentum $p_\phi = xp_y - yp_x$ is conserved and the hamiltonian becomes

$$H = \frac{1}{2}\frac{p_r^2}{m} + U(r) + \frac{p_\phi^2}{2mr^2}, \qquad (109)$$

Once the system is reduced to one degree of freedom, Hamilton equations allow to solve it evaluating some integral.

Thus the name integrable systems.

Poincare sections and invariant tori

There is no a priori analytical method to know before hand if there are additional constants of motion.

One of the advantages of the canonical transformations is that they are a tool to transform one system into another with the hope of identifying the system better.

Even for integrable systems, numerical analysis may be of little help, because even with the numerical

solution giving us the trajectories, these could be too complicated.

For example, if we have a harmonic potential in each coordinate

$$U_x = \frac{1}{2}\omega_x^2 x^2, \qquad U_y = \frac{1}{2}\omega_y^2 y^2 \tag{110}$$

If ν is rational we get Lissajous figures that could be complicated, and if ν is irrational then the phase space becomes dense with the trajectories.

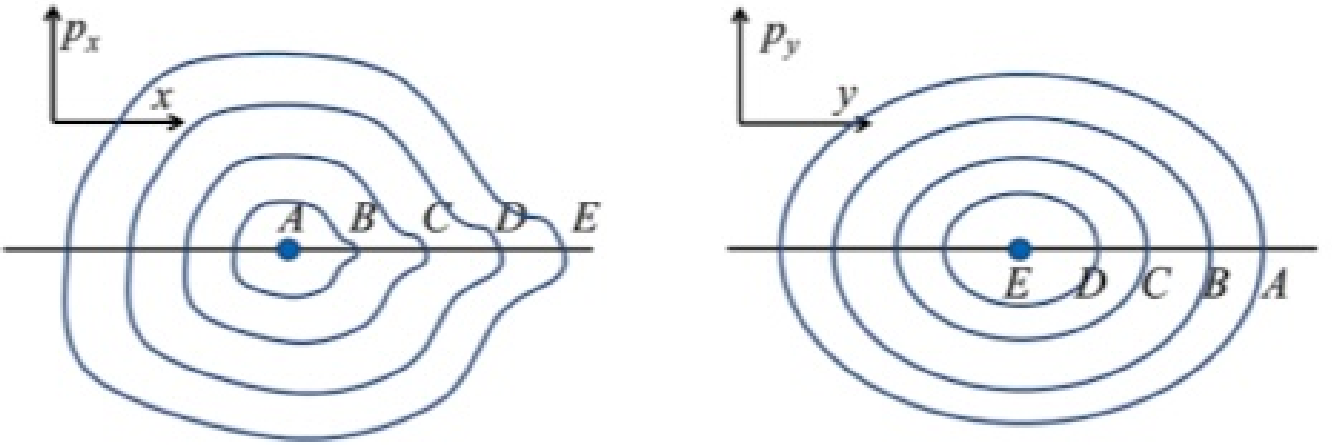
A very astute technique is a method that Poincare figured out.

Consider the separable potential of the double oscillator.

Motion for each coordinate are independent.

So we may plot the trajectories in two planes of the phase space, (x, p_x) and (y, p_y).

Remember that the particle is confined in its motion.



If U_x and U_y have each a simple minimum, trajectories will be like in the figure.

The particle moves in each of these projections of the 4-D space.

Each curve corresponds to one level of energy.

If we change the initial conditions, and one of the curves gets bigger, then the other shrinks because $E = E_x + E_y$.

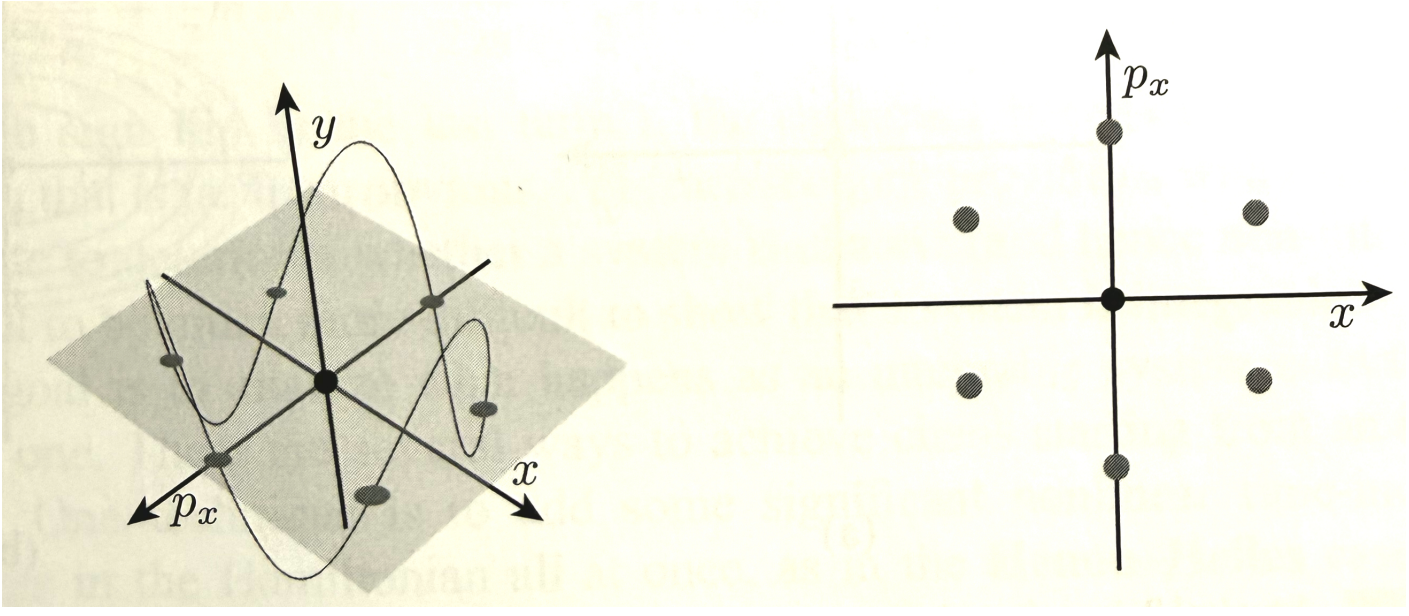
In each plane there is a maximum curve that corresponds to all the energy associated to that coordinate.

These curves are the intersections of the 2 dimensional manifold where the trajectories live in phase space, with subspaces (x, p_x) and (y, p_y).

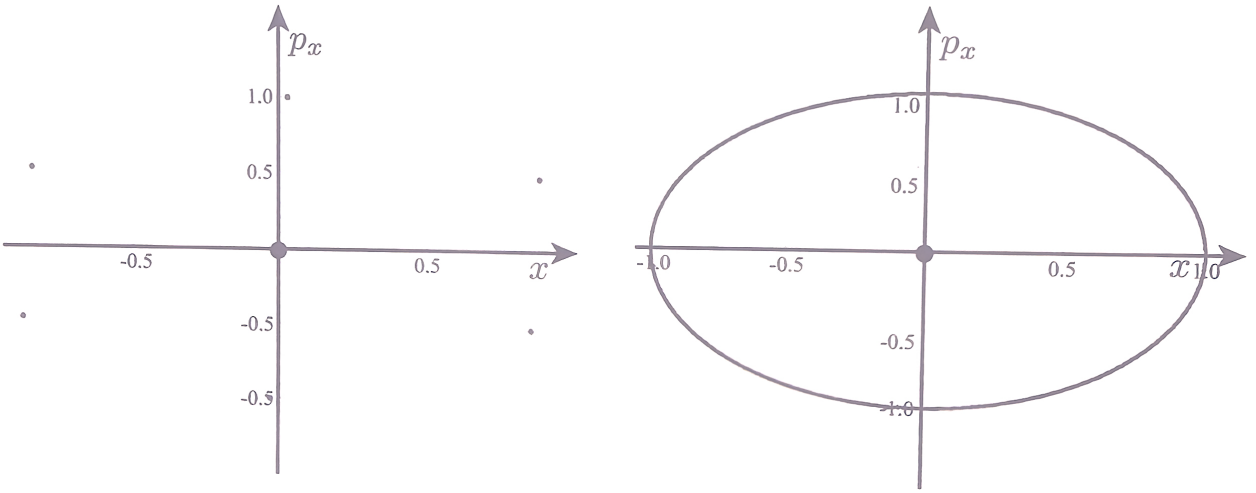
For this reason these 2-D plots are called surface sections or also **Poincare maps**.

Closed curves in this sections are a manifestation of the system's integrability.

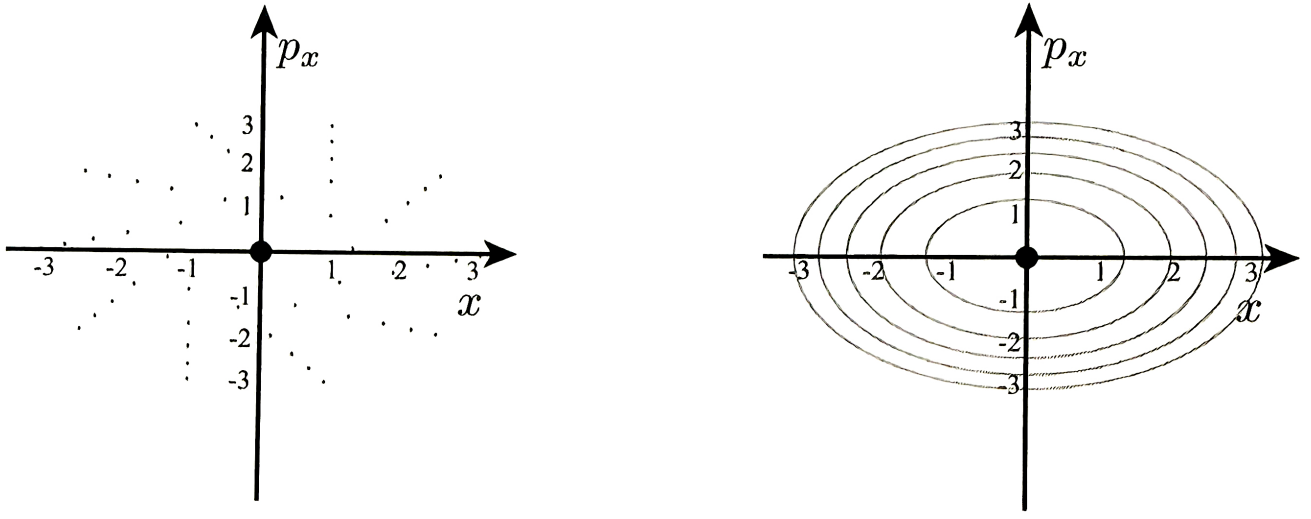
The curves themselves are called invariant tori.



Left the trajectory in x, y, p_x phase space and right the $x - p_x$ Poincare map for the double oscillator.



Left:Poincare map for the double oscillator for ν rational. Right: ν irrational.



Poincare map for the double oscillator with different initial conditions; ν rational on the left and ν irrational on the right.

We can do this for any system, starting with the trajectory corresponding to a given initial condition.

What we do is look at a given coordinate, i.e. x, and let's say every time the particle goes through 0 with $p_x > 0$ we sketch the position of the particle in the plane (y, p_y).

This is a "stroboscopic" visualization of the trajectory.

If the periods of motion in x and y are such the winding number ν is rational these observations would

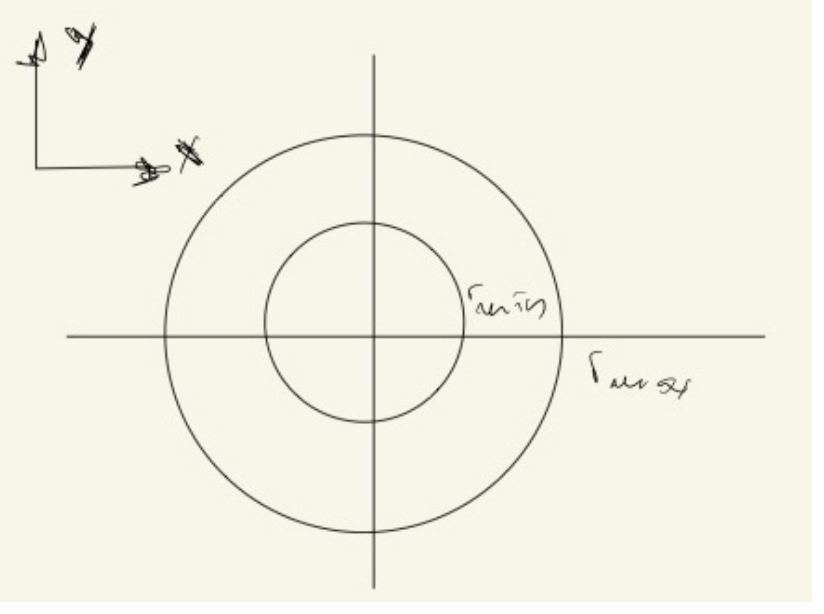
give isolated points in the plane, because the trajectories closed up and will transverse the same places again.

If the periods are incommensurable (ν irrational) the observations will give us solid curves.

In this manner we can study the topology in phase space, that corresponds to any Hamiltonian just analyzing the system trajectories.

Let's look at the case of a central potential to study the general behavior of an integrable hamiltonian.

For given values of the energy and the angular momentum, the radial motion of a particle is confined between r_{min} and r_{max}.



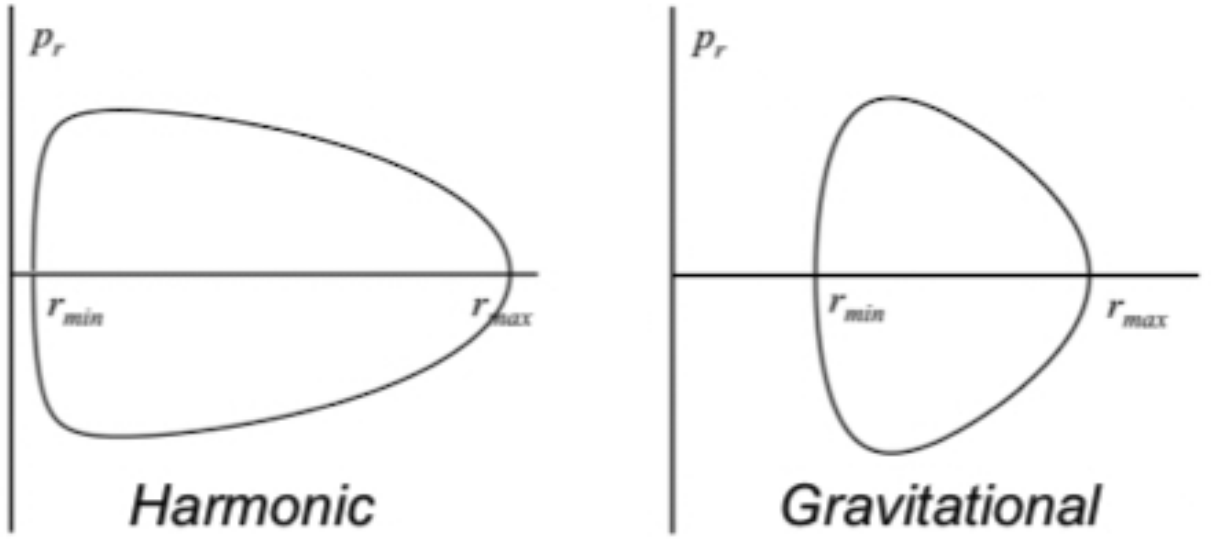
These are the solutions of

$$E - U(r) - \frac{p_\phi^2}{2mr^2} = 0 \qquad (111)$$

In the (x, y) plane, these two radii define an annulus.

Conservation of angular momentum allows to calculate for each value of r the momentum p_r which can only take two values:

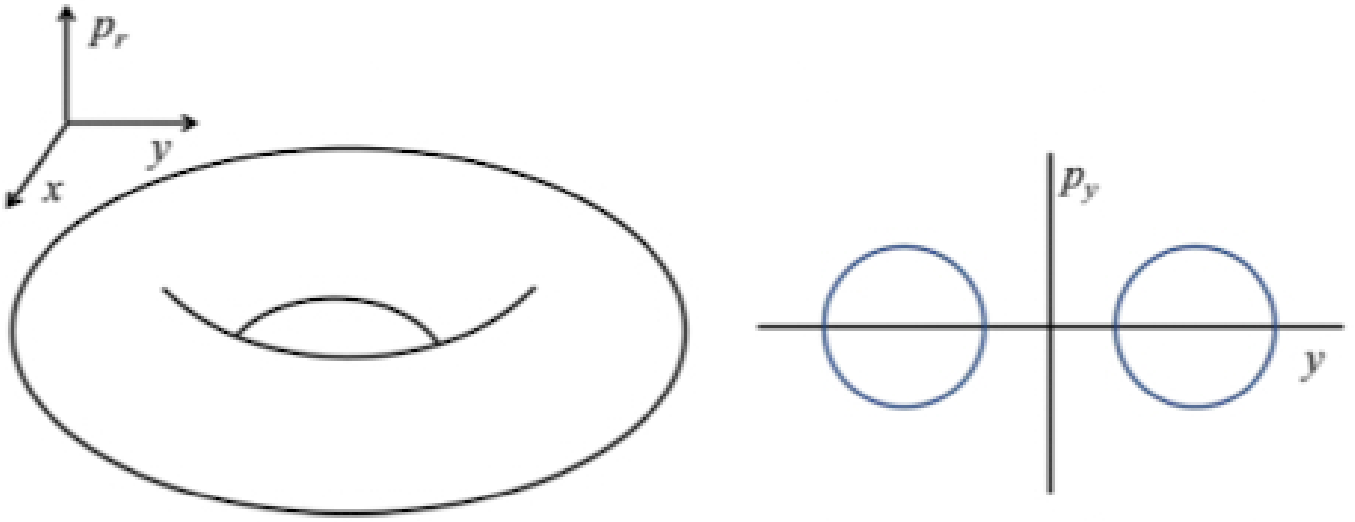
$$p_r = \pm\sqrt{2E - 2U(r) - \frac{p_\phi^2}{2mr^2}} \qquad (112)$$



These values of the momentum define a 2-dimensional manifold in (x, y, p_r) where the trajectory is confined.

This surface has the topology of a torus. If we take a section of the torus, when $x = 0$, in the plane $(y, p_r \equiv p_y)$ we get two closed curves.

If we keep the energy but change the angular momentum (i.e. initial velocity), the size of the torus changes and the size of the Poincare maps changes as well.



For each value of the constants of motion these trajectories are confined to these surfaces which are the invariant tori.

When a small nonlinear perturbation is added, the system becomes nearly integrable.

Many invariant tori slightly bend and persist as **KAM tori**.

KAM: Kolmogorov, Arnol'd and Moser.

For each value of the energy there are many invariant tori.

Each one corresponds to a value of the other conserved quantity (i.e. angular momentum in a central potential).

In general a Poincare section will look like in the figure.

The marked points are the fixed elliptic points which correspond to trajectories that repeat themselves exactly after some period.

There are similar to the minima of effective potentials.

Around them are nested tori.

In the case of the central potential there is a unique elliptic point that corresponds to a circular orbit of a given energy (E in the figure).

The tori around it have the same energy but different angular momenta.

There could be more than one elliptic point, and separating the closed tori there curves called **separatrices** (singular separatrix).

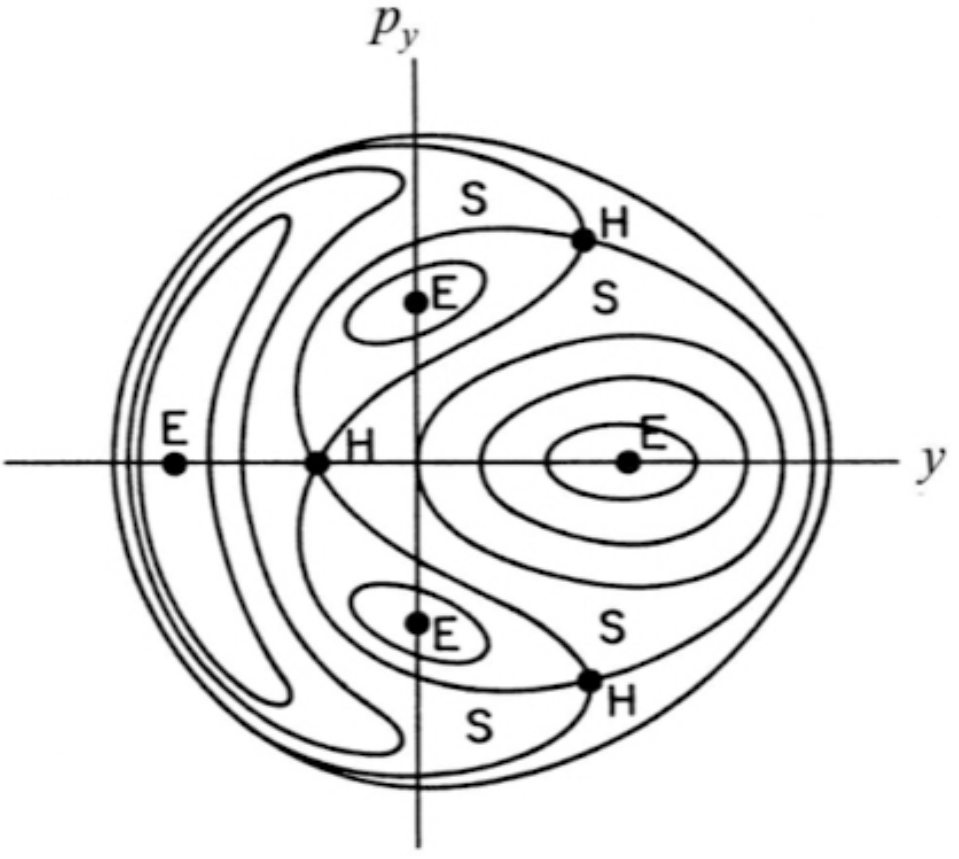
Separatrices (S in the figure) are boundary trajectories that divide phase space into qualitatively different regions of motion.

Where they cross we have fixed points called hyperbolics (H in the figure).

These give an “stroboscopic vision” of the periodic trajectories.

But, while the E points correspond to trajectories stable against small perturbations, points H correspond to trajectories which are stable against perturbations in one direction (asymptotic to a hyperbolic trajectory).

But they are unstable in the other direction. In summary, unstable trajectories.


p_y
y
S
H
E
S
E
H
E
E
S
S
H

There is a theorem by Chirikov that shows that the existence of these intersections between stable and unstable manifolds is the necessary condition for a hamiltonian system to exhibit chaotic behavior.

What happens to with the tori of integrable systems when experiencing a perturbation that destroys integrability?

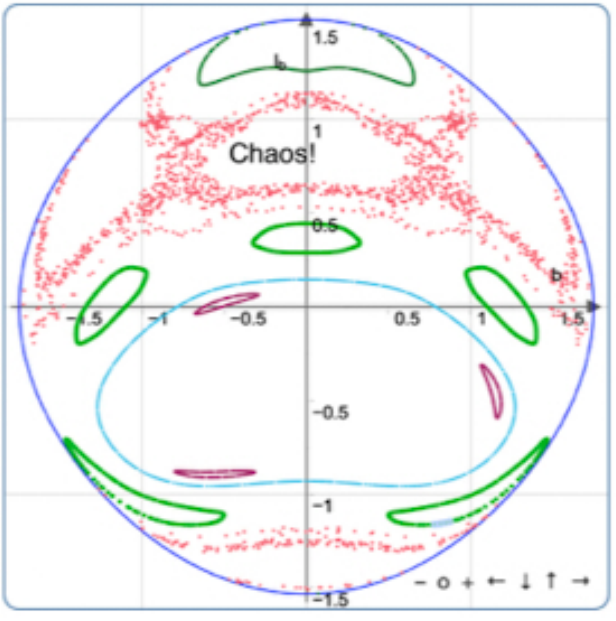
For a small perturbation, the majority of the tori around the elliptic points become distorted, but they still keep the nested tori topology.

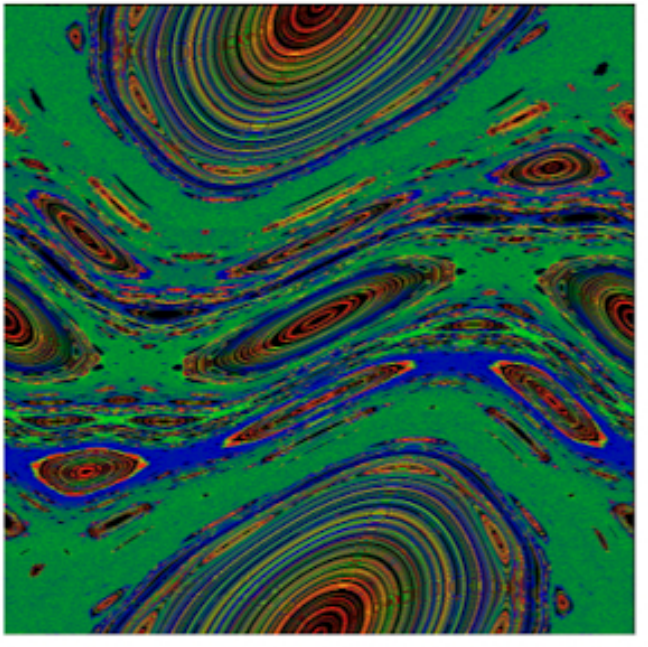
This is what the theorem KAM (by Kolomogorov, Arnold and Moser) says.

What is the meaning of majority?

It means that in adjacent regions of phase space trajectories become chaotic, producing Poincare sections that seem randomly distributed points (no fixed points, or regular curves).

And inside these chaotic regions we find other elliptic points with their nested small tori, giving an infinitely intricate topology, hierarchical and fascinating.





The corresponding trajectories are called strange attractors.