Classical Mechanics 2026
Homework 9

Exercise 1.

Integrate the logistic equation to find $x(t)$:

$$\dot{x} = rx(t)\Big(1 - \frac{x(t)}{K}\Big), \tag{1}$$

where r and K are positive constants. (Hint: use $y = 1/x$.)

Exercise 2.

Plot the variant of the supercritical pitchfork given by equation (11):

$$\dot{x} = rx + x^3. \tag{2}$$

This is called the subcritical pitchfork. What is the most important difference with the supercritical one?

Exercise 3.

And if we look at this other one:

$$\dot{x} = rx + x^3 - x^5 \tag{3}$$

what is the difference now?

Exercise 4.

Solve the linear first order system of differential equations.

$$\begin{aligned} \dot{\epsilon} &= \frac{\partial f}{\partial x}\Big|_{x^*,y^*}\epsilon + \frac{\partial f}{\partial y}\Big|_{x^*,y^*}\eta, \\ \dot{\eta} &= \frac{\partial g}{\partial x}\Big|_{x^*,y^*}\epsilon + \frac{\partial g}{\partial y}\Big|_{x^*,y^*}\eta, \end{aligned} \tag{4}$$

trying the solution

$$\epsilon = c_1 e^{\lambda t}, \qquad \eta = c_2 e^{\lambda t}, \tag{5}$$

Determine c_1, c_2, λ.

Exercise 5.

Find the critical points of the system and calculate the Jacobian to determine that one is a center and the other one a saddle point.

$$\begin{aligned} \dot{\theta} =& \nu, \\ \dot{\nu} =& -\sin\theta, \end{aligned} \tag{6}$$

Exercise 6.
Assume $x_1(t)$ and $x_2(t)$ are solutions of

$$p(t)\ddot{x} + q(t)\dot{x} + r(t)x(t) = 0 \tag{7}$$

Show that then a linear combination of them is a solution as well, but if we look instead at the equation

$$p(t)\ddot{x} + q(t)\dot{x} + r(t)\sqrt{x(t)} = 0 \tag{8}$$

Then a linear combination of $x_1(t)$ and $x_2(t)$ is not a solution.

Exercise 7.

Motion for a damped harmonic oscillator can be modeled:

$$\ddot{x} + \mu\dot{x} + \omega_0^2 x = 0 \quad \mu > 0. \tag{9}$$

Find the fixed points and plot the in phase space (x, y) where $y = \dot{x}$ for the following cases:
a) underdamped: $0 < \mu < 2\omega_0$..
b) over-damped: $\mu > 2\omega_0$.
c) critical damping: $\mu = 2\omega_0$.

Exercise 8.
Double well

Study the 1-degree of freedom system:

$$H(p, q) = \frac{p^2}{2m} + aq^2 + bq^4. \tag{10}$$

Find and classify the fixed points, and plot the trajectories in phase space. Consider the cases in which a and b are positives and negatives.

Exercise 9.

Find and classify all equilibrium points for the following dynamical system:

$$\dot{x} = -x + x^3, \qquad \dot{y} = -2y. \tag{11}$$

Plot the phase diagram of the complete system by analyzing the linearized behavior near equilibrium and nullclines.

Exercise 10.

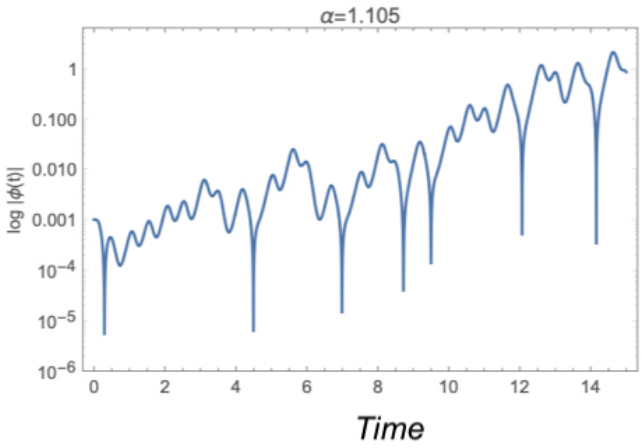
Analyze the following system

$$\begin{aligned} \dot{x} &= -y + ax(x^2 + y^2), \\ \dot{y} &= x + ay(x^2 + y^2). \end{aligned} \tag{12}$$

where a is a parameter. Show that the center of coordinates is a stable spiral for $a < 0$ and an unstable one for $a > 0$. (Hint: remember $x^2 + y^2 = r^2$).

Exercise 11.

The figure shows the separate evolution of two identical pendulums, where their respective initial conditions differ in 10^{-3} radians. Estimate their Lyapunov exponent and determine if motion is chaotic or not.



Assume that you know the initial position of this system with a precision of 10^{-6} radians, and do you have to do a prediction of the pendulum position with a precision of $1/100$ of a radian. What is the maximum time that allows you to improve this precision? And if you manage to improve the initial position up to 10^{-9}. What is the new time horizon to have the same precision? how much has it improved?