PHYS 5310 - CLASSICAL MECHANICS - 2026

HOMEWORK 8

Exercise 1.
Prove the conservation of momentum and angular momentum of a system of particles if the Hamiltonian of the system doesn't change: a) under an infinitesimal translation.b) under an infinitesimal rotation.

Exercise 2.
Determine the Poisson brackets formed from the Cartesian components of the momentum and the angular momentum of a particle. Show all your calculations.

Exercise 3.
Find the Hamiltonian for a single particle in Cartesian, Elliptical and Parabolic coordinates.

Exercise 4.
Show that $[M_z, \phi] = 0$ where ϕ is a spherically symmetric scalar function of the coordinates and momenta.

Exercise 5.
Show that $[M_z, \vec{f}] = \hat{n} \times \vec{f}$ where $\vec{f}$ is a vectorial function of the coordinates and momenta of a particle and $\hat{n}$ is a unit vector in the direction of z.

Exercise 6.
Following the variational principle

$$\delta \int \sqrt{2m(E-U)} dl = 0$$

derive the differential equation of the path followed by a particle.

Exercise 7.
Prove the properties of the Poisson brackets described in formulas (34)-(37) Lesson 8.

$$[f, g] = -[g, f]$$

$$[f, c] = 0$$

if c is a constant. Also if we have to functions f_1, f_2,

$$[f_1 + f_2, g] = [f_1, g] + [f_2, g]$$

$$[f_1 \times f_2, g] = f_1[f_2, g] + f_2[f_1, g]$$

If we want to take the time derivative of the Poisson bracket

$$\frac{\partial}{\partial t}[f, g] = \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right]$$

Exercise 8.
Prove formulas

$$[q_i, q_k] = 0 \qquad [p_i, p_k] = 0 \qquad [p_i, q_k] = \delta_{ik}$$

Exercise 9.

Prove Jacobi's identity

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

by direct calculation.

Exercise 10.
Consider the simple harmonic oscillator with Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \tag{1}$$

If we were to effectuate a coordinate transformation from the coordinates (q, p) to new coordinates (Q, P) to obtain a Hamiltonian that would look like

$$\widetilde{H} = \omega P \tag{2}$$

1) What would the Hamilton's equations be when using the Hamiltonian in equation (2)?
2) Solve these equations obtaining $Q(t)$ and $P(t)$.
3) Show that the exact form needed for the transformations to yield the Hamiltonian (1) expressed as in (2) are:

$$q(Q, P) = \sqrt{\frac{2P}{m\omega}}\sin Q, \qquad p(Q, P) = \sqrt{2m\omega P}\cos Q \tag{3}$$

4) Show that the proposed transformation is canonical calculating the corresponding Poisson brackets for the new coordinates.

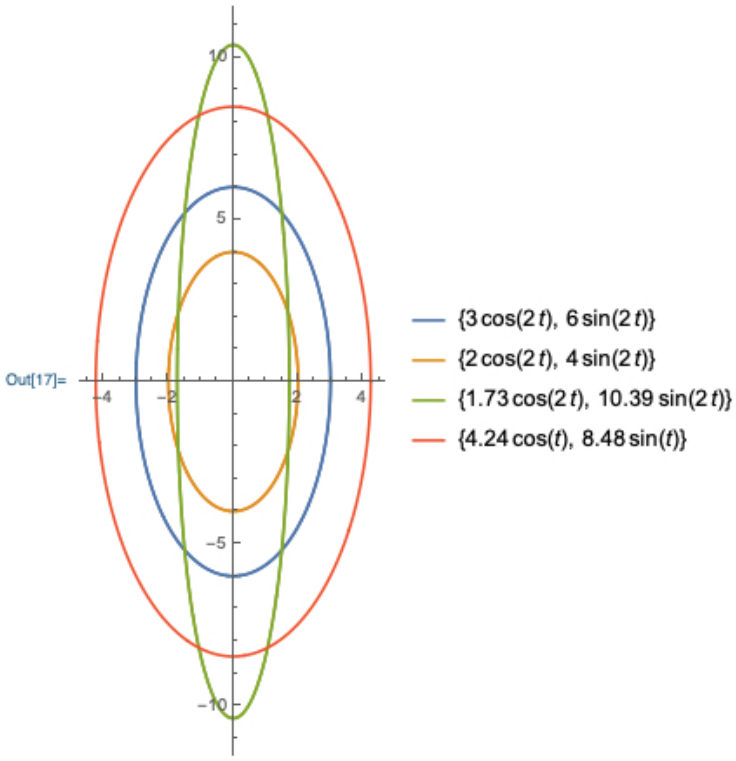


Figure 1: Phase space

Exercise 11.
The plot in Figure 1 represents the behavior in phase space of 4 different Harmonic Oscillators. The following are the physical parameters of the systems (assume that the units are the appropriate ones, the horizontal axis is q, the vertical one is p):

	m	ω	q_0
blue	1	2	3
yellow	1	2	2
green	3	2	$\sqrt{3}$
red	2	1	$\sqrt{18}$

m is the mass of the system involved, ω is the frequency and q_0 is the amplitude. 1) What adiabatic invariant can you identify in these different phase plots? 2) What is the physical meaning and how are these systems related?

Exercise 12.

Calculate the integral in

$$I_r = 2\frac{1}{2\pi}\int_{r_{min}}^{r_{max}} \sqrt{2m\big(E+\frac{\alpha}{r}\big) - \frac{M^2}{r^2}}dr \tag{4}$$

showing that

$$I_r = -M + \alpha\sqrt{\frac{m}{2|E|}} \tag{5}$$

To this effect:
1) Extract r^2 from the integral to get

$$I_r = \frac{1}{\pi}\int_{r_{min}}^{r_{max}} \frac{dr}{r}\sqrt{Ar^2+Br+C} \tag{6}$$

Integrate using the result from table of integrals. 2) To calculate r_min and r_max remember that these are the roots of the equation

$$U_{eff} = -|E| \tag{7}$$

where

$$U_{eff} = -\frac{\alpha}{r} + \frac{M^2}{2mr^2} \tag{8}$$

Note: the energy has to be negative for having a close path.