HW 1 Solution

$$r = \sqrt{x^2+y^2}$$

Ex 1

$$F_x = m\ddot{x}$$

$$\frac{1}{\cancel{2}}\frac{\cancel{2}x}{\sqrt{x^2+y^2}}$$

$$F_y = m\ddot{y}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$\begin{pmatrix} F_r \\ F_\theta \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

$$\begin{pmatrix} F_r \\ F_\theta \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{\left(\sqrt{1+\frac{x^2}{y^2}}\right)x^2} & \frac{1}{\sqrt{1+\frac{x^2}{y^2}}}\frac{1}{x} \end{pmatrix} \begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix}$$

$$\begin{pmatrix} F_r \\ F_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{\sin\theta}{r} & -\frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} m\frac{d^2x(r,\theta)}{dt^2} \\ m\frac{d^2y(r,\theta)}{dt^2} \end{pmatrix}$$

The only purpose of this rather tedious exercise is to realize the importance of a covariant treatment in physics, like the Lagrangian formalism.

Exercise 2 Double pendulum

The Lagrangian is

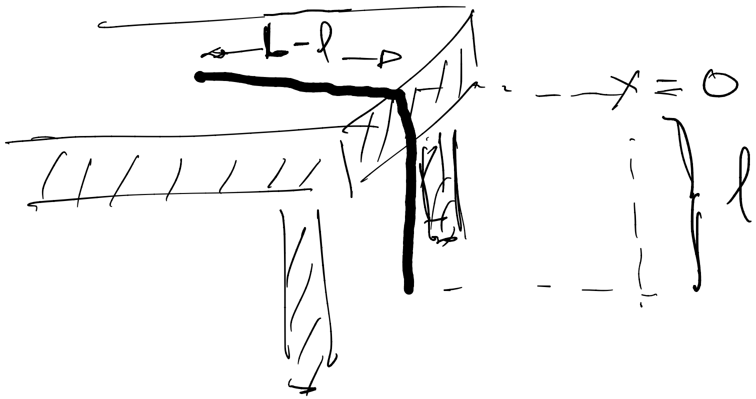
$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2 l_2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1\dot{\phi}_2 \cos(\phi_1-\phi_2)$$

$$+ (m_1+m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2$$

The equations of motion can be obtained noticing that we have two generalized coordinates ϕ_1 and ϕ_2.

$$\left.\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_1}\right) - \frac{\partial L}{\partial \phi_1} = 0 \\ &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_2}\right) - \frac{\partial L}{\partial \phi_2} = 0 \end{aligned}\right\}$$

Exercise 3



We choose x as a generalized coordinate.
The weight of a segment of length l of rope is
$F = \mu g x$. So $U = -\int F\,dx \rightarrow U = \frac{\mu g x^2}{2}$.
While the weight of $L-l$ is balanced
out by normal reaction force of the table.
But for the kinetic energy friction is 0.

So $T = \frac{1}{2}\sum m_i v_i^2 = \frac{1}{2} M v^2 = \frac{1}{2} M \dot{x}^2$

$T = \frac{1}{2}\mu L \dot{x}^2$ ← all the rope moves with the same velocity!

Then $L = \frac{1}{2}\mu L \dot{x}^2 + \frac{\mu g x^2}{2}$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\mu L \ddot{x} - \mu g x = 0$$

$$\ddot{x} - \omega^2 x = 0 \qquad \omega = \sqrt{g/L}$$

$\rightarrow x(t) = A\cosh(\omega t) + B\sinh(\omega t)$

$\dot{x}(0) = 0 \qquad x(0) = l$

$\rightarrow A = l \quad \& \quad B = 0$

So $\boxed{x(t) = l\cosh\left(\sqrt{g/L}\, t\right)}$

Exercise 4

$$T = \frac{1}{2} m \dot{x}^2 \qquad U = -\frac{k x^2}{2}$$

$$L = T - U = \frac{1}{2} m \dot{x}^2 + \frac{k x^2}{2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} - kx = 0$$

$$\ddot{x} = \omega^2 x \qquad \omega = \sqrt{\frac{k}{m}}$$

$$x = A\cos\omega t + B\sin\omega t$$

A & B determined by initial

$x(0)$ & $\dot{x}(0)$

Exercise 5

1) The Lagrangian is additive.
For two systems A & B far apart so they do not interact $L \rightarrow = L_A + L_B$

2) E-L equations do not change if the Lagrangian is multiplied by a constant.

3) A Lagrangian is only defined up to an additive total time derivative of a function of coordinates and time.

Exercise 6

Assume $x=0$ when $t=0$ then $C=0$

If $t=t_a$ when $x=a$

Then $a = A t_a^2 + B t_a$

And $B = \left(\frac{a}{t_a} - A t_a\right)$

So $x(t) = A t^2 + \left(\frac{a}{t_a} - A t_a\right) t$

Action is

$$S = \int_0^{t_a} L(x, \dot{x},) dt = \int_0^{t_a} \left(\frac{m}{2}\dot{x}^2 + Fx\right) dt$$

$$= \frac{m A^2 t^3}{6} + \frac{m a^2}{2 t_a} - \frac{F A t^3}{6} + \frac{F a t_a}{2}$$

$\frac{\partial S}{\partial A} = 0$ minimum of action

$$\frac{2m A t^3}{6} = \frac{F t^3}{6} \rightarrow A = \frac{F}{2m}$$

$$x(t) = \frac{F}{2m} t^2 + \left(\frac{a}{t_a} - \frac{F t_a}{2m}\right) t$$

Exercise 7

You only have to calculate

$$x = r\cos\phi \qquad dx = dr\cos\phi - r\sin\phi\, d\phi$$
$$y = r\sin\phi \qquad dy = dr\sin\phi + r\cos\phi\, d\phi$$
$$z = z \qquad dz = dz$$

$$dx^2 + dy^2 = dr^2\cos^2\phi + r^2\sin^2\phi\, d\phi^2 - 2r\sin\phi\cos\phi\, dr\, d\phi$$
$$+ dr^2\sin^2\phi + r^2\cos^2\phi\, d\phi^2 + 2r\sin\phi\cos\phi\, dr\, d\phi$$
$$= dr^2 + r^2 d\phi^2$$

So $dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\phi^2 + dz^2 = dl^2$

and $\dfrac{dl^2}{dt^2} = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2$

Similarly it is easy to prove

with $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

, etc.