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Lesson 8: The Canonical Equations

Mario C Díaz

Hamilton's equations

Sometimes it is more convenient to use generalized coordinates and generalized momenta instead of generalized coordinates and velocities like we were doing so far: $q_i, \dot{q}_i \to q_i, p_i$. We assuming that i takes values ranging from 1 to n the number of degrees of freedom corresponding to the physical system under consideration.

The transformation of coordinates that it is required can be done utilizing a *Legendre transformation*. Starting with the the Lagrangian we can assume that a small change in the generalized coordinates dq_i and the generalized velocities $d\dot{q}_i$ will induce a differential transformation in the Lagrangian function which to first order is:

$$\begin{aligned} dL &= \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \\ &= \sum_i \dot{p}_i dq_i + \sum_i p_i d\dot{q}_i \end{aligned} \tag{1}$$

where the last equation is the result of substituting $\frac{\partial L}{\partial \dot{q}_i} = p_i$ just by definition of momentum and $\dot{p}_i = \frac{\partial L}{\partial q_i}$ using Lagrange equations. But we also notice that $\sum_i p_i d\dot{q}_i$ can be written as a total differential:

$$\sum_i p_i d\dot{q}_i = d\left(\sum_i p_i \dot{q}_i\right) - \sum_i \dot{q}_i dp_i \tag{2}$$

Using this result in (1)

$$dL = \sum_i \dot{p}_i dq_i + d\left(\sum_i p_i \dot{q}_i\right) - \sum_i \dot{q}_i dp_i \tag{3}$$

from which we obtain

$$d\left(\sum_i p_i \dot{q}_i - L\right) = -\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i \tag{4}$$

or defining a new function which it is called the Hamiltonian[1]:

$$H(p,q,t)=\sum_i p_i\dot{q}_i - L \tag{5}$$

which is precisely what we define as the energy of the system in the Lagrangian formalism. From this definition (4) reads:

$$dH=-\sum_i \dot{p}_i dq_i+\sum_i \dot{q}_i dp_i \tag{6}$$

From where it is simple to read the Hamilton equations:

$$\dot{q}_i=\frac{\partial H}{\partial p_i} \qquad \dot{p}_i=-\frac{\partial H}{\partial q_i} \tag{7}$$

Notice that these are a set of $2n$ first order differential equations replacing the n set of second order differential equations which are the Euler-Lagrange equations. These equations of motion in the Hamiltonian formalism are also called the “canonical equations”.

Conservation of Energy

The total time derivative of the Hamiltonian is:

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}+\sum_i \frac{\partial H}{\partial q_i}\dot{q}_i+\sum_i \frac{\partial H}{\partial p_i}\dot{p}_i=\frac{\partial H}{\partial t} \tag{8}$$

The last equation is because, due to Hamilton equations, $\dot{q}_i=\frac{\partial H}{\partial p_i}, \dot{p}_i=-\frac{\partial H}{\partial q_i}$ and then the second term in (8) is equal to the third one. We can see then that if the Hamiltonian doesn’t depend explicitly on time,

$$\frac{dH}{dt}=0 \tag{9}$$

and the energy is conserved.

Like the Lagrangian, the Hamiltonian involve parameters. These parameters characterize the physical system itself. Let λ be such a parameter.Then

$$dL=\sum_i \dot{p}_i dq_i+\sum_i p_i d\dot{q}_i+\left(\frac{\partial L}{\partial \lambda}\right)d\lambda \tag{10}$$

[1] Sir William Rowan Hamilton (4 August 1805 – 2 September 1865) was an Irish mathematician, physicist and astronomer.

on the other side the Hamiltonian

$$dH = -\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i - \left(\frac{\partial L}{\partial \lambda}\right) d\lambda \tag{11}$$

which implies that

$$\left(\frac{\partial H}{\partial \lambda}\right)\bigg|_{p,q} = -\left(\frac{\partial L}{\partial \lambda}\right)\bigg|_{q,\dot{q}} \tag{12}$$

where p, q and $q, \dot{q}$ remain constant in each case.

We can arrive at the same conclusion in a different manner. Let the Lagrangian for a given physical system be of the form $L = L_0 + L'$ where L' is a small correction to L_0. Then we can define a new Hamiltonian for our system H' as in,

$$H = H_0 + H' \tag{13}$$

such that

$$H'\Big|_{p,q} = -L'\Big|_{q,\dot{q}} \tag{14}$$

and

$$\left(\frac{\partial H}{\partial t}\right)\bigg|_{p,q} = -\left(\frac{\partial L}{\partial t}\right)\bigg|_{q,\dot{q}} \tag{15}$$

The Routhian

Let's suppose there are generalized coordinates and velocities $\rho, \xi, \dot{\rho}, \dot{\xi}$ and we want to transform them to $\rho, \xi, p_\rho, \dot{\xi}$ where p_ρ is the conjugate momentum to ρ. In that case we will have

$$dL = \frac{\partial L}{\partial q}dq + \frac{\partial L}{\partial \dot{q}}d\dot{q} + \frac{\partial L}{\partial \xi}d\xi + \frac{\partial L}{\partial \dot{\xi}}d\dot{\xi} = \dot{p}dq + pd\dot{q} + \frac{\partial L}{\partial \xi}d\xi + \frac{\partial L}{\partial \dot{\xi}}d\dot{\xi} \tag{16}$$

From where we can rearrange

$$d(L - p\dot{q}) = \dot{p}dq + pd\dot{q} + \frac{\partial L}{\partial \xi}d\xi + \frac{\partial L}{\partial \dot{\xi}}d\dot{\xi} \tag{17}$$

And we can define the Routhian as the following function

$$R(q, p, \xi, \dot{\xi}) = p\dot{q} - L \tag{18}$$

from where we obtain

$$dR = -\dot{p}dq + \dot{q}dp - \frac{\partial L}{\partial \xi}d\xi - \frac{\partial L}{\partial \dot{\xi}}d\dot{\xi} \tag{19}$$

and the following equations of motion:

$$\begin{aligned} \dot{q} = \frac{\partial R}{\partial p}, \qquad\qquad & \dot{p} = -\frac{\partial R}{\partial q} \\ \frac{\partial L}{\partial \xi} = -\frac{\partial R}{\partial \xi}, \qquad\qquad & \frac{\partial L}{\partial \dot{\xi}} = -\frac{\partial R}{\partial \dot{\xi}} \end{aligned} \tag{20}$$

with the addition of the second order differential equation

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\xi}}\right) = \frac{\partial R}{\partial \xi} \tag{21}$$

In conclusion the Routhian is a Hamiltonian for the ρ coordinate and a Lagrangian for ξ. If we want to write an expression for the energy of the system in terms of the Routhian

$$E = \dot{q}\frac{\partial L}{\partial \dot{q}} + \dot{\xi}\frac{\partial L}{\partial \dot{\xi}} - L = p\dot{q} + \dot{\xi}\frac{\partial L}{\partial \dot{\xi}} - L \tag{22}$$

which gives explicitly in terms of R

$$E = R - \dot{\xi}\frac{\partial R}{\partial \dot{\xi}} \tag{23}$$

Why bothering defining a function such that the Routhian? When a coordinate is cyclic it might be a convenient way of finding the equations of motion. For example, if ρ is a cyclic coordinate the Routhian depends only on $p, \xi, \dot{\xi}$. Using (20) we inmediatly find that

$$\dot{p} = -\frac{\partial R}{\partial q} = 0 \tag{24}$$

from where we can quickly conclude that p is a constant. Also in this case

$$\frac{d}{dt}\left(\frac{\partial R(p,\xi,\dot{\xi})}{\partial \dot{\xi}}\right) = \frac{\partial R(p,\xi,\dot{\xi})}{\partial \xi} \tag{25}$$

where it is clear that only ξ is involved.
Once ξ is obtained we can solve

$$\dot{q} = \frac{\partial R}{\partial p} \tag{26}$$

by direct integration.

Poisson Brackets

Let f(p, q, t) be some function of the coordinates, momentum and time, Its total time derivative is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_k \left(\frac{\partial f}{\partial q_k}\dot{q}_k + \frac{\partial f}{\partial p_k}\dot{p}_k \right) \tag{27}$$

$\dot{q}_k$ and $\dot{p}_k$ are given by the canonical equations $\dot{q}_i = \frac{\partial H}{\partial p_i}$ $\dot{p}_i = -\frac{\partial H}{\partial q_i}$. so we can substitute

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial H}{\partial q_k} \right) \tag{28}$$

or

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [H, f] \tag{29}$$

where

$$[H, f] = \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial H}{\partial q_k} \right) \tag{30}$$

$[H, f]$ is called the Poisson bracket of h and f.

If the quantity f is an integral of motion it fulfills $df/dt = 0$ so in (29) we obtain

$$\frac{\partial f}{\partial t} + [H, f] = 0 \tag{31}$$

If the function f is additionally independent explicitly of time,

$$[H, f] = 0 \tag{32}$$

The Poisson bracket of the Hamiltonian and the function which is an integral of motion is zero. This definition can be extended to two functions

$$[f, g] = \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k} \right) \tag{33}$$

Properties of the Poisson bracket

Following the definition it is easy to prove the following properties

$$[f, g] = -[g, f] \tag{34}$$

$$[f, c] = 0 \tag{35}$$

if c is a constant. Also if we have to functions f_1, f_2,

$$[f_1 + f_2, g] = [f_1, g] + [f_2, g] \tag{36}$$

$$[f_1 \times f_2, g] = f_1[f_2, g] + f_2[f_1, g] \tag{37}$$

If we want to take the time derivative of the Poisson bracket (33)

$$\frac{\partial}{\partial t}[f, g] = \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right] \tag{38}$$

If one of the functions f and g is one of the momenta or coordinates the Poisson bracket reduces to a partial derivative

$$[f, q_k] = \frac{\partial f}{\partial p_k} \tag{39}$$

$$[f, p_k] = \frac{\partial f}{\partial q_k} \tag{40}$$

From where we can obtain

$$[q_i, q_k] = 0 \qquad [p_i, p_k] = 0 \qquad [p_i, q_k] = \delta_{ik} \tag{41}$$

If we have three functions

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0 \tag{42}$$

This is called the Jacobi's identity. It can be proved by direct calculation (brute force). But it's quite important in defining the nature of the Poisson brackets. We will prove it in the following manner. Let's define by introducing a square matrix $\mathbf{J}$ made up of Poisson brackets. i.e. equation (41) can be written in the following manner,

$$J_{ij} = [\boldsymbol{\eta}, \boldsymbol{\eta}] \tag{43}$$

where $[\boldsymbol{\eta}, \boldsymbol{\eta}]$ is a matrix whose element i, j is $[\eta_i, \eta_j]$ We can introduce the following functions associated to partial derivatives respect to the corresponding canonical variable

$$f_i \equiv \frac{\partial f}{\partial \eta_i}, \qquad\qquad g_{ij} \equiv \frac{\partial g}{\partial \eta_i \partial \eta_j} \tag{44}$$

Using this notation the Poisson bracket of f and g can be expressed as

$$[f, g] = f_i J_{ij} g_j \tag{45}$$

where, of course, J_{ij} is the i, j element of the matrix J_{ij} defined by (43). Considering now in (42) the first Poisson bracket:

$$[f, [g, h]] = f_i J_{ij} [g, h]_j = f_i J_{ij} (g_k J_{kl} hl)_j. \tag{46}$$

Due to the fact that the elements J_{ij} are constants, the derivative with respect to η does not affect them and we then have

$$[f, [g, h]] = f_i J_{ij} (g_k J_{kl} h_{lj} + g_{kj} J_{kl} h_l). \tag{47}$$

where remember that the two subindices in (f, g, h) refer to the corresponding second partial derivatives respect to the canonical variables. And the other two Poisson brackets can be obtained from (47) by cyclic permutation of f, g, h. There will be six terms in total each, each being a quadruple sum over dummy indices i, j, k, adn l. Consider now the term in (47) which involves a second derivative of h

$$J_{ij} J_{kl} f_i g_k h_{lj}, \tag{48}$$

It is not difficult to see that the only other term involving a second derivative of h will appear in the second double bracket in the Jacobi's identity (42)

$$[g, [h, f]] = g_k J_{kl} (h_j J_{ji} f_i)_l. \tag{49}$$

where the term containing the second derivative in h is

$$J_{ij} J_{kl} f_i g_k h_{jl}, \tag{50}$$

but comparing (48) and (50) $h_{lj} = h_{jl}$ (the order of the derivatives does not change the results). The sum of both terms is then

$$(J_{ij} + J_{ji}) J_{kl} f_i g_k h_{lj} = 0, \tag{51}$$

due to the antisymmetry of $\mathbf{J}$. Th eother four terms are cyclic permutations and the same process performed above can be applied to the second derivatives of f and g, verifying Jacobi's identity.

Poisson's theorem

If f and g are integrals of motion, their Poisson bracket is also an integral of motion,

$$[f, g] = constant \tag{52}$$

Proof

Let's apply the Jacobi's identity (42) using as a function h, the Hamiltonian of the system itself, ie. $h = H$.

$$[H, [f, g]] + [f, [g, H]] + [g, [H, f]] = 0 \tag{53}$$

but due to the fact that f and g are integrals of motion

$$[H, g] = 0 = [H, f] \tag{54}$$

So in (53) the only surviving term is

$$[H, [f, g]] = 0 \tag{55}$$

which indicates that $[f, g]$ is indeed an integral of motion as well.

An alternative and instructive way of also proving this result is by directly calculating the time derivative of the Poisson bracket using (29):

$$\frac{d}{dt}[f, g] = \frac{\partial}{\partial t}[f, g] + [H, [f, g]] \tag{56}$$

and now using (38)

$$\frac{\partial}{\partial t}[f, g] = \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right] \tag{38}$$

we get

$$\begin{aligned}
\frac{d}{dt}[f, g] &= \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right] - [f, [g, H]] - [g, [H, f]] \\
&= \left[\frac{\partial f}{\partial t} + [H, f], g\right] + \left[f, \frac{\partial g}{\partial t} + [H, g]\right] \\
&= \left[\frac{df}{dt}, g\right] + \left[f, \frac{dg}{dt}\right]
\end{aligned} \tag{57}$$

and clearly if f and g are integrals of motion it proves the theorem.

The action as a function of the coordinates

Let's consider the action as a function of the upper limit of integration t_2, i.e.

$$S = \int_{t_1}^{t_2} L dt \tag{58}$$

Looking at a variation of the action

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q dt \tag{59}$$

The second term in the above equation, the integral, is zero, because of Lagrange's equations. For the first term in the sum we put $\delta q(t_1) = 0$ and $\delta q(t_2) = \delta q$. Then (59) becomes

$$\delta S = p \delta q \tag{60}$$

or assuming n degrees of freedom we get

$$\delta S = \sum_i p_i \delta q_i \tag{61}$$

which implies

$$\frac{\partial S}{\partial q_i} = p_i \tag{62}$$

On the other hand from the definition of action

$$\frac{dS}{dt} = L \tag{63}$$

an calculating explicitly

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i} \dot{q}_i = \frac{\partial S}{\partial t} + \sum_i p_i \dot{q}_i \tag{64}$$

From where we have

$$\frac{\partial S}{\partial t} = \frac{dS}{dt} - \sum_i p_i \dot{q}_i = L - \sum_i p_i \dot{q}_i = -H \tag{65}$$

where we use (63) and the definition of the Hamiltonian (5). From here we can write

$$dS = \sum_i p_i dq_i - H dt \tag{66}$$

Let's assume now that the change in S occurs due to variation in the coordinates and time at the beginning and end of the path

$$dS = \sum_i p_i^{(2)} dq_i^{(2)} - H^{(2)} dt^{(2)} - \sum_i p_i^{(1)} dq_i^{(1)} - H^{(1)} dt^{(1)} \tag{67}$$

The only possible motion is the one for which dS is a perfect differential (i.e.$dS = 0$). The principle of action imposes restrictions on the range of possible motions. Interesting enough the Hamilton's equations can be derived from defining the action

$$S = \int \left(\sum_i p_i dq_i - H dt \right) \tag{68}$$

we then have

$$\delta S = \int \left(\delta p dq + p d\delta q - \frac{\partial H}{\partial q} \delta q dt - \frac{\partial H}{\partial p} \delta p dt \right) \tag{69}$$

Integrating by parts the second term we get

$$\delta S = \int \delta p \left(dq - \frac{\partial H}{\partial p} dt \right) + [p, \delta q] - \int \delta q \left(dp + \frac{\partial H}{\partial q} \right) dt \tag{70}$$

at the limits of integration we have $\delta q = 0$. We are left with two integrals where the integrands will have to vanish independently because δp and δq vary independently and arbitrarily. For this to be the case we need

$$dq = \frac{\partial H}{\partial p} dt \qquad\qquad dp = -\frac{\partial H}{\partial q} dt \tag{71}$$

and dividing by dt are the Hamilton's equations.

Maupertuis' Principle

There is a simplified form of the Principle of least action which is useful if we are more interested in finding the path to be followed by a physical system than the precise position as a function of time.
Let's assume that the energy is conserved, and consequently both the Hamiltonian and the Lagrangian do not depend explicitly of time.

$$H(p, q) = E = constant \tag{72}$$

We will also allow for a variation of the final time we are considering for a given path (instead of t_0, t we will consider $t_0, t + \delta t$. We have

$$\delta S = -H \delta t \tag{73}$$

We are comparing not all possible virtual motions but only those which conserve the energy,

$$\delta S + E\delta t = 0 \tag{74}$$

Writing the action as

$$S = \int \left(\sum_i p_i dq_i - H dt \right) \tag{75}$$

From where we obtain

$$S = \int \sum_i p_i dq_i - E(t - t_0) \tag{76}$$

The first term above is called the abbreviated action

$$S_0 = \int \sum_i p_i dq_i \tag{77}$$

The abbreviated action fulfills

$$\delta S_0 = 0 \tag{78}$$

It has a minimum with respect to all paths which satisfy the law of conservation of energy and pass through the final point at any instant.
In order to use this principle, the momenta need to be expressed as a functions of q and dq.

$$p_i = \frac{\partial}{\partial \dot{q}} L(q, \dot{q}) \tag{79}$$

and

$$E = E(q, \dot{q}) \tag{80}$$

Expressing dt in terms of q_i and dq_i in (80) and substituting in (79) we will have the momenta in terms of q_i and dq_i with E as a parameter. The variational principle obtained in this manner is called the Maupertuis' Principle.

In the case that the Lagrangian takes its original form

$$L = \frac{1}{2} \sum_{ik} a_{ik}(q) \dot{q}_i \dot{q}_k - U(q) \tag{81}$$

where the momenta are

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \sum_k a_{ik}(q)\dot{q}_k \tag{82}$$

and

$$E = \frac{1}{2}\sum_{ik} a_{ik}(q)\dot{q}_i\dot{q}_k + U(q) \tag{83}$$

where the last equation gives

$$dt = \sqrt{\frac{\sum_{ik} a_{ik}(q)dq_i dq_k}{2(E-U)}} \tag{84}$$

In order to obtain an expression where we use E as a parameter in (77) we substitute in $\sum_i p_i dq_i = \sum_{ik} a_{ik}\frac{dq_k}{dt}dq_i$ and then

$$S_0 = \int \sqrt{2(E-U)\sum_{ik} a_{ik}(q)dq_i dq_k} \tag{85}$$

In the case of a single particle we have

$$T = \frac{1}{2}m\left(\frac{dl}{dt}\right)^2 \tag{86}$$

from where

$$S_0 = \int \sqrt{2(E-U)mdl^2} \tag{87}$$

then Maupertuis

$$\delta S_0 = \delta \int \sqrt{2(E-U)m}dl = 0 \tag{88}$$

and if If the particle is free $U = 0$ and

$$\sqrt{2Em}\int dl = 0 \tag{89}$$

which shows that the straight line is the shortest path in the absence of U. Returning to (76)

$$S = \int \sum_i p_i dq_i - E(t-t_0) \qquad (76)$$

we will consider a variation respect to E

$$\delta S = \frac{\partial S_0}{\partial E}\delta E - (t - t_0)\delta E - E\delta t \tag{90}$$

Substituting in (74)

$$\delta S + E\delta t = 0 \qquad\qquad (74)$$

we get

$$\frac{\partial S_0}{\partial E} = t - t_0 \tag{91}$$

where $S_0 = \int \sqrt{2(E-U)\sum_{ik} a_{ik}(q)dq_i dq_k}$ and taking the derivative we get

$$\int \sqrt{\frac{\sum_{ik} a_{ik}(q)dq_i dq_k}{2(E-U)}} = t - t_0 \tag{92}$$

which is the integral of (84)

$$dt = \sqrt{\frac{\sum_{ik} a_{ik}(q)dq_i dq_k}{2(E-U)}}$$

Together with the equation for the path it determines the motion.

Canonical Transformations

In the Lagrangian formalism we can have s generalized coordinates $q_i \to Q_k$ such that $Q_i = Q_i(q,t)$. In the Lagrangian formalism ew have $2s$ generalized coordinates which can be transformed

$$q, p \to Q_i = Q_i(p,q,t), P_i = P_i(p,q,t). \tag{93}$$

But the equations do not retain their canonical form under all possible transformations like (93). What are the conditions that need to be satisfied so that with new coordinates P, Q

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i}, \qquad\qquad P_i = -\frac{\partial H'}{\partial Q_i} \tag{94}$$

for some H'.

One class of transformations is particularly important. These are called canonical transformations. We know that Hamilton's equations can be obtained from

$$\delta \int (\sum_i p_i dq_i - H dt) = 0 \tag{95}$$

If the new coordinates P, Q satisfy the Hamilton's Principle

$$\delta \int (\sum_i P_i dQ_i - H' dt) = 0 \tag{96}$$

Then the difference between (95) and (96) should just be a function of coordinates and time

$$\sum p_i dq_i - H dt = \sum P_i dQ_i - H' dt + dF \tag{97}$$

which we can write as

$$dF = \sum p_i dq_i - \sum P_i dQ_i + (H' - H) dt \tag{98}$$

from where we obtain

$$p_i = \frac{\partial F}{\partial q_i}, \qquad P_i = \frac{\partial F}{\partial Q_i}, \qquad H' = H + \frac{\partial F}{\partial t} \tag{99}$$

F is called the generating function. We assume $F = F(q, Q, t)$. When F is known equation (99) give the relationship between p, q and P, Q as well as the new Hamiltonian. In some situations it may be more convenient to give F in terms not of q, Q but of q, P.

In these cases we apply a Legendre transformation

$$d(F + \sum P_i Q_i) = \sum_i p_i dq_i + \sum Q_i dP - i + (H' - H) dt \tag{100}$$

If we call $F + \sum P_i Q_i = \Phi(q, P, t)$ we have

$$p_i = \frac{\partial \Phi}{\partial q_i}, \qquad Q_i = \frac{\partial \Phi}{\partial P_i}, \qquad H' = H + \frac{\partial \Phi}{\partial t} \tag{101}$$

Similarly we can obtain canonical transformations involving generating functions of p, Q or p, P. Always $H' - H$ is the time derivative of the generating function. If f does not depend of the time $H' = H$. Notice also that with this formalism we can have $Q_i = p_i$ and $P_i = -q_i$. This is just an exchange of names for the variables. In this context of arbitrariness of the meaning of p and q, they are called 5canonically conjugated quantities.

Example

Consider the simple harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \tag{102}$$

where $\omega = \sqrt{\frac{k}{m}}$. Let's consider

$$F_1(q, Q, t) = qQ \tag{103}$$

From where using the relationships established in equation (101) we get

$$p_k = \frac{\partial F_1}{\partial q_k}, \qquad P_k = \frac{\partial F_1}{\partial Q_k}, \qquad H' = H + \frac{\partial F_1}{\partial t} \tag{104}$$

and consequently we find $p = Q$ and $P = -q$ and

$$H' = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 + 0 = \frac{Q^2}{2m} + \frac{1}{2}m\omega^2 p^2 \tag{105}$$

and

$$\dot{Q} = \frac{\partial H'}{\partial P} = m\omega^2 P, \qquad \text{and} \qquad \dot{P} = \frac{\partial H'}{\partial Q} = -\frac{Q}{m} \tag{106}$$

which shows that Q, P is physically equivalent to q, p.

General Theorem about Poisson brackets

The conditions relating canonically conjugated variables can be expressed in terms of the Poisson brackets.
If we have the Poisson brackets associated to two functions f and g calculated as derivatives respect to the coordinates p, q and P, Q then

$$[f, g]_{p,q} = [f, g]_{P,Q} \tag{107}$$

This can be demonstrated by direct calculation using the canonical transformations explicitly. But it can also be demonstrated in the following manner. We notice that in equations (99) and (101) the time appears as a parameter. This makes it sufficient to prove (107) for functions that do not depend explicitly on time.
Let's think that g is the Hamiltonian of some system. Then following (29)

$$[f, g]_{p,q} = -\frac{df}{dt} \tag{108}$$

In the equation above df/dt depends only of the physical system it describes, of course, and not on the variables utilized to describe it. Consequently the Poisson bracket $[f, g]$ does not change when transforming from one set of canonical coordinates to the other. Using together (41) and (108) we get that P, Q need to satisfy

$$[Q_i, Q_k]_{p,q} = 0 \qquad [P_i, P_k]_{p,q} = 0 \qquad [P_i, Q_k]_{qp} = \delta_{ik} \tag{109}$$

if the transformation $q, p \to P, Q$ is canonical.

Notice that $p(t), q(t)$ can be thought as they evolve with time as a set of canonical transformations, where S the action is the generating function.

Liouville's Theorem

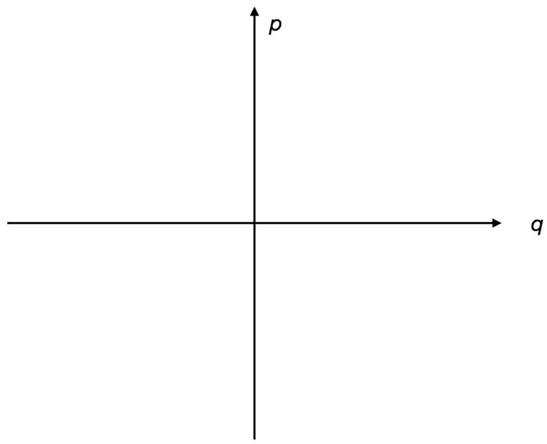

Figure 1: The phase space

The *phase space* is made up of the generalized coordinates and momenta. For a system with n degrees of freedom it is $2n$ dimensional. A point in the phase space corresponds to a value of the generalized coordinates like the position of a physical system and the corresponding generalized momenta at the same time (ie. a state of the system).
As the system evolves with time the result of its motion represented as a curve in the phase space is called its phase path.

$$d\Gamma = dq_1 \times dq_2 \times .... \times dq_{n-1} \times dq_n \times dp_1 \times dp_2 \times .... \times dp_{n-1} \times dp_n \tag{110}$$

is an element of volume in phase space.

Liouville's Theorem states that

$$\int d\Gamma = constant \tag{111}$$

i.e. the integral $\int d\Gamma$ over some region of the phase space is invariant under canonical transformations.

For example let's consider variables p, q which through a canonical transformation are replaced by P, Q. We want to prove that

$$\int d\Gamma = dq_1 \times ...dq_n \times dp_1 \times ... \times dp_n = \int dQ_1 \times ...dQ_n \times dP_1 \times ... \times dP_n \tag{112}$$

Due to the transformation of coordinates

$$\int dQ_1 \times ...dQ_n \times dP_1 \times ... \times dP_n = \int \frac{\partial(Q_1...Q_n P_1...P_n)}{\partial(q_1...q_n p_1...p_n)} \times dq_1 \times ...dq_n \times dp_1 \times ... \times dp_n \tag{113}$$

where

$$D = \frac{\partial(Q_1...Q_n P_1...P_n)}{\partial(q_1...q_n p_1...p_n)} \tag{114}$$

is the Jacobian of the transformation (the determinant of the matrix of the coordinate transformation that would take q_i, p_i to Q_i, P_i). All we need to prove then is that $D = 1$.

We can divide numerator and denominator in (114) by $\partial(q_1...q_n P_1...P_n)$

$$D = \frac{\frac{\partial(Q_1...Q_n P_1...P_n)}{\partial(q_1...q_n P_1...P_n)}}{\frac{\partial(q_1...q_n p_1...p_n)}{\partial(q_1...q_n P_1...P_n)}}$$

As we can see in this Jacobian $P_1...P_n$ appears both in the numerator and denominator of the numerator. The same happens with $q_1...q_n$ in the denominator. so we get

$$D = \frac{\frac{\partial(Q_1...Q_n)}{\partial(q_1...q_n)}_{P=constant}}{\frac{\partial(p_1...p_n)}{\partial(P_1...P_n)}_{q=constant}}$$

The Jacobian in the numerator is by definition, a determinant of order n whose ith row and kth column is $\partial Q_i/\partial q_k$. Using (101) we can write it in terms of the generating function Φ as $\partial Q_i/\partial q_k = \partial^2\Phi/\partial q_k \partial P_i$. Following the same line of thought we get in the denominator that the ik element of the determinant is $\partial^2\Phi/\partial q_i \partial P_k$. This means that the two determinants only differ in the interchange of row for columns. Being equal it shows then that $D = 1$, completing the proof.

Significance of the Liouville's theorem:

If we have a system of particles undergoing an evolution over time determined by the equations of motion, as they move in phase space the volume they occupy remains unchanged

$$\int d\Gamma = constant$$

The Hamilton-Jacobi equation

When we considered before the action as a function of coordinates and time -as in (65)- it was shown that

$$\frac{\partial S}{\partial t} + H(p, q, t) = 0 \tag{115}$$

and

$$\frac{\partial S}{\partial q_i} = p_i \tag{116}$$

We could generalize this result in this manner

$$\frac{\partial S}{\partial t} + H\left(q_1...q_n; \frac{\partial S}{\partial q_1}...\frac{\partial S}{\partial q_n}; t\right) = 0 \tag{117}$$

(117) is called the Hamilton-Jacobi equation. Recapping, we have so far studied the Lagrange's equations, the canonical equations and now the Hamilton-Jacobi equations. The first one is a second order system of differential equations. The last two are first order differential equations.

The independent variables in the H-J equations are the time and the coordinates. For a system with n degrees of freedom a complete integral must contain $n+1$ arbitrary constants. There will be one constant of integration A such that

$$S = f(t, q_1...q_n; \alpha_1...\alpha_n) + A \tag{118}$$

where $\alpha_1...\alpha_n$ and A are the $n+1$ arbitrary constants of integration.

We will make a canonical transformation from q, p using $f(t, q, \alpha)$ as a generating function and $\alpha_1...\alpha_n$ as the new momenta.
Let the new coordinates be $\beta_1...\beta_n$. The generating function then will satisfy

$$p_i = \frac{\partial f}{\partial q_i}, \qquad \beta_i = \frac{\partial f}{\partial \alpha_i}, \qquad H' = H + \frac{\partial f}{\partial t} \tag{119}$$

And f will also satisfy the Hamilton-Jacobi equations (117), so

$$H' = H + \frac{\partial f}{\partial t} = H + \frac{\partial S}{\partial t} = 0 \tag{120}$$

we also have $\dot{\alpha}_i = 0$ due to $\alpha_1...\alpha_n$ being constants and $\dot{\beta}_i = 0$ due to $\beta_i = \frac{\partial f}{\partial \alpha_i}$.
The n coordinates q_i can be expressed in terms of the time and the $2n$ constants α and β. This give the general integral of motion. The H-J method continues in this manner. After finding

$$S = f(t, q_1...q_n; \alpha_1...\alpha_n) + A \tag{121}$$

we calculate

$$\frac{\partial S}{\partial \alpha_i} = \beta_i \tag{122}$$

which gives the coordinates as functions of time and of the $2n$ arbitrary constants. Then the momenta are

$$p_i = \frac{\partial S}{\partial q_i} \tag{123}$$

If the system is conservative, the H-J takes a simpler form

$$S = S_0(q) - Et \tag{124}$$

and substituting in

$$\frac{\partial S}{\partial t} + H\left(q_1 ... q_n; \frac{\partial S}{\partial q_1} ... \frac{\partial S}{\partial q_n}; t\right) = 0 \tag{125}$$

gives the solution for the abbreviated action S_0

$$H\left(q_1 ... q_n; \frac{\partial S_0}{\partial q_1} ... \frac{\partial S_0}{\partial q_n}\right) = E \tag{126}$$

Separation of variables

Let's assume that one coordinate, i.e. q_1, and $\partial S/\partial q_1$ appear in a particular functional form

$$\phi\left(q_1, \frac{\partial S}{\partial q_1}\right)$$

which does not involve any of the other coordinates or derivatives or the time, i.e.

$$\Phi\left(q_i, t, \frac{\partial S}{\partial q_i}, \frac{\partial S}{\partial t}, \phi\left(q_1, \frac{\partial S}{\partial q_1}\right)\right) = 0 \tag{127}$$

where $i \neq 1$ We look for a solution of the form

$$S = S'(q_i, t) + S_1(q_1), \tag{128}$$

We can plug this in (127) we get

$$\Phi\left(q_i, t, \frac{\partial S'}{\partial q_i}, \frac{\partial S'}{\partial t}, \phi\left(q_1, \frac{\partial S_1}{\partial q_1}\right)\right) = 0 \tag{129}$$

If we assume that we already found (128) when substituted in (129) it must become an identity valid for any value of q_1. But if this is true then ϕ must be a constant. So we can split (129) in two separate equations:

$$\phi\left(q_1, \frac{dS}{dq_1}\right) = \alpha_1 \tag{130}$$

and

$$\Phi\left(q_i, t, \frac{\partial S'}{\partial q_i}, \frac{\partial S'}{\partial t}, \alpha_1\right) = 0 \tag{131}$$

where α_1 is an arbitrary constant. (130) is an ordinary differential equation (ODE) and S_1 is then obtained by integration. (131) involves fewer variables. If we could follow the same procedure for all the q_i then we would get a full set of ODEs. As a result the H-J equation could be solved by quadratures.

If we are dealing with a conservative system we have to separate n variables (assuming n the degrees of freedom of the system).
Once the separation is complete we shall get

$$S = \sum_k S_k(q_k, t, \alpha_1, \alpha_2 ... \alpha_n) - E(\alpha_1, \alpha_2 ... \alpha_n)t \tag{132}$$

S_k depends on only one coordinate. The energy E as a function of $\alpha_1, \alpha_2 ... \alpha_n$ is obtained by substituting $S_0 = \sum S_k$ in

$$H\left(q_1 ... q_n, \frac{\partial S_0}{\partial q_1} ... \frac{\partial S_0}{\partial q_n}\right) = E \tag{133}$$

A particular case is when one variable is cyclic, ie. let's assume q_1; q_1 then is not appearing in the Hamiltonian nor in the H-J equations. In that case the function $\phi(q_1, \partial S/\partial q_1) \to \partial S/\partial q_1$ and then $S_1 = \alpha_1 q_1$ and we obtain

$$S = S'(q_i, t) + \alpha_1 q_1 \tag{134}$$

and α_1 is just the value of $p_1 = \partial S/\partial q_1$ which corresponds to the cyclic coordinate. If we have a conservative system then $-Et$ corresponds to the "cyclic" coordinate t.

Examples

a) Spherical coordinates (r, θ, ϕ)

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + U(r, \theta, \phi) \tag{135}$$

We can choose

$$U(r, \theta, \phi) = a(r) + \frac{b(\theta)}{r^2} + \frac{c(\phi)}{r^2 \sin^2\theta} \tag{136}$$

where $a(r), b(\theta), c(\phi)$ are arbitrary functions. We can simplify the problem a bit by assuming that the potential doe snot depend on ϕ.

$$U(r, \theta, \phi) = a(r) + \frac{b(\theta)}{r^2} \tag{137}$$

The H-J equations for S_0 are

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial r}\right)^2 + a(r) + \frac{1}{2mr^2}\left[\left(\frac{\partial S_0}{\partial \theta}\right)^2 + 2mb(\theta)\right] + \frac{1}{2mr^2 \sin^2\theta}\left(\frac{\partial S_0}{\partial \phi}\right)^2 = E \tag{138}$$

Since ϕ is cyclic we look for

$$S_0 = p_\phi \phi + S_1(r) + S_2(\theta) \tag{139}$$

which plugged in the equation (138) gives

$$\left(\frac{dS_2}{d\theta}\right)^2 + 2mb(\theta) + \frac{p_\phi^2}{\sin^2\theta} = \beta \tag{140}$$

$$\frac{1}{2m}\left(\frac{dS_1}{dr}\right)^2 + a(r) + \frac{\beta}{2mr^2} = E \tag{141}$$

Integrating we get

$$S = -E + p_\phi \phi + \int \sqrt{\beta - 2mb(\theta) - \frac{p_\phi^2}{\sin^2\theta}}\, d\theta + \int \sqrt{2m\left(E - a(r)\right) - \frac{\beta}{r^2}}\, dr \tag{142}$$

where the arbitrary constants are p_ϕ, β and E.

b) parabolic coordinates

We can transform from cylindrical to parabolic coordinates $(\rho, \phi, z) \rightarrow (\xi, \eta, \phi)$, where the transformation is performed through

$$z = \frac{1}{2}(\xi - \eta) \qquad\qquad \rho = \sqrt{\xi\eta} \tag{143}$$

where $0 < \xi < \infty$ and $0 < \eta < \infty$. Surfaces of constant ξ and η are paraboloids of revolution with z the axis of symmetry. Notice that in (143) we can identify a relationship with the spherical radius r

$$\xi = r + z \qquad\qquad \eta = r - z \tag{144}$$

If we calculate the Lagrangian in cylindrical coordinates

$$L = \frac{1}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - U(\rho, \phi, z) \tag{145}$$

Substituting ξ, η, ϕ we get

$$L = \frac{1}{8}m(\xi + \eta)\left(\frac{\dot{\xi}^2}{\xi} + \frac{\dot{\eta}^2}{\eta}\right) + \frac{1}{2}m\xi\eta\dot{\phi}^2 - U(\xi, \eta, \phi) \tag{146}$$

where the moment are

$$p_\xi = \frac{1}{4}m(\xi + \eta)\frac{\dot{\xi}}{\xi} \tag{147}$$

$$p_\eta = \frac{1}{4}m(\xi + \eta)\frac{\dot{\eta}}{\eta} \tag{148}$$

$$p_\phi = m\xi\eta\dot{\phi} \tag{149}$$

and the Hamiltonian

$$H = \frac{2}{m}\frac{\xi p_\xi^2 + \eta p_\eta^2}{\xi + \eta} + \frac{p_\phi^2}{2m\xi\eta} + U(\xi, \eta, \phi) \tag{150}$$

A physically interesting potential is

$$U = \frac{a(\xi) + b(\eta)}{\xi + \eta} = \frac{a(r + z) + b(r - z)}{2r} \tag{151}$$

So

$$\frac{2}{m(\xi + \eta}\left[\xi\left(\frac{\partial S_0}{\partial \xi}\right)^2 + \eta\left(\frac{\partial S_0}{\partial \eta}\right)^2\right] + \frac{1}{2m\xi\eta}\left(\frac{\partial S_0}{\partial \phi}\right)^2 + \frac{a(\xi) + b(\eta)}{\xi + \eta} = E \tag{152}$$

ϕ can be separated as $p_\phi \phi$ multiplying by $m(\xi+\eta)$ and then we would get

$$2\xi\left(\frac{\partial S_0}{\partial \xi}\right)^2 + ma(\xi) - mE\xi + \frac{p_\phi^2}{2\xi} + 2\eta\left(\frac{\partial S_0}{\partial \eta}\right)^2 + mb(\eta) - mE\eta + \frac{p_\phi^2}{2\eta} = 0 \tag{153}$$

and making

$$S_0 = p_\phi \phi + S_1(\xi) + S_2(\eta) \tag{154}$$

we get

$$2\xi\left(\frac{dS_1}{d\xi}\right)^2 + ma(\xi) - mE\xi + \frac{p_\phi^2}{2\xi} = \beta \tag{155}$$

$$2\eta\left(\frac{dS_2}{d\eta}\right)^2 + mb(\eta) - mE\eta + \frac{p_\phi^2}{2\eta} = \beta \tag{156}$$

Integration gives

$$S = -Et + p_\phi \phi + \int \sqrt{\frac{m}{2}E + \frac{\beta}{2\xi} - \frac{ma(\xi)}{2\xi} - \frac{p_\phi^2}{4\xi^2}}d\xi + \int \sqrt{\frac{m}{2}E - \frac{\beta}{2\eta} - \frac{mb(\eta)}{2\eta} - \frac{p_\phi^2}{4\eta^2}}d\eta \tag{157}$$

where the arbitrary constants are p_ϕ, β and E.

b) Elliptic coordinates

These are ξ, η, ϕ defined

$$\rho = \sigma\sqrt{(\xi^2-1)(1-\eta^2)}, \qquad z = \sigma\xi\eta \tag{158}$$

where σ is a parameter and the range for ξ, η is $1 < \xi < \infty, \qquad -1 < \eta < 1$. If A_1 and A_2 are points on the z axis they have

$$\begin{aligned} z =& \pm\sigma \\ r_1 =& \sqrt{(z-\sigma)^2+\rho^2} \\ r_2 =& \sqrt{(z+\sigma)^2+\rho^2} \end{aligned} \tag{159}$$

Using (158)

$$r_1 = \sigma(\xi-\eta) \qquad r_2 = \sigma(\xi+\eta) \tag{160}$$

$$\xi = \frac{r_2+r_1}{2\sigma} \qquad \eta = \frac{r_2-r_1}{2\sigma} \tag{161}$$

The Lagrangian can be easily transformed to elliptical coordinates from cylindrical ones:

$$\frac{1}{2}m\sigma^2(\xi^2-\eta^2)\left(\frac{\dot{\xi}^2}{\xi^2-1}+\frac{\dot{\eta}^2}{1-\eta^2}\right)+\frac{1}{2}m\sigma^2(\xi^2-1)(1-\eta^2)\dot{\phi}^2-U(\xi,\eta,\phi) \tag{162}$$

And from it the Hamiltonian is

$$H=\frac{1}{2m\sigma^2(\xi^2-\eta^2)}\left[(\xi^2-1)p_\xi^2+(1-\eta^2)p_\eta^2+\left(\frac{1}{\xi^2-1}+\frac{1}{1-\eta^2}\right)p_\phi^2\right]+U(\xi,\eta,\phi) \tag{163}$$

A physically interesting case for U is

$$U=\frac{a(\xi)+b(\eta)}{\xi^2-\eta^2}=\frac{\sigma^2}{r_1r_2}\left\{a\left(\frac{r_2+r_1}{2\sigma}\right)+b\left(\frac{r_2-r_1}{2\sigma}\right)\right\} \tag{164}$$

$a(\xi)$ and $b(\eta)$ are arbitrary functions. The result is

$$\begin{aligned}S=-Et+p_\phi\phi+\int\sqrt{2m\sigma^2E+\frac{\beta-2m\sigma^2a(\xi)}{\xi^2-1}-\frac{p_\phi^2}{(\xi^2-1)^2}}d\xi+\\ \int\sqrt{2m\sigma^2E-\frac{\beta+2m\sigma^2b(\eta)}{1-\eta^2}-\frac{p_\phi^2}{(1-\eta^2)^2}}d\eta\end{aligned} \tag{165}$$

Note

Surfaces of constant ξ are ellipsoids which in canonical form can be expressed:

$$\frac{z^2}{\sigma^2\xi^2}+\frac{\rho^2}{\sigma^2(\xi^2-1)}=1 \tag{166}$$

Their foci are A_1 and A_2.
The surfaces of constant η are hyperboloids

$$\frac{z^2}{\sigma^2\eta^2}+\frac{\rho^2}{\sigma^2(1-\eta^2)}=1 \tag{167}$$

Also with foci A_1 and A_2.

Example

Find a complete integral of the H-J equation for a motion of a particle in a field $U=\alpha/r-Fz$ (a Coulomb and a uniform field). Find a conserved function of the coordinates and the momenta that is specific to this motion.

Solution

The field is

$$U = \frac{a(\xi) + b(\eta)}{\xi + \eta} = \frac{a(r+z) + b(r-z)}{2r} \tag{168}$$

where the coordinates are the parabolic ones

$$\xi = r + z \qquad\qquad \eta = r - z \tag{169}$$

and

$$\begin{aligned} a(\xi) &= \alpha - \frac{1}{2}F\xi^2 \\ b(\eta) &= \alpha + \frac{1}{2}F\eta^2 \end{aligned} \tag{170}$$

and then the solution is given by (157)

$$\begin{aligned} S = -Et + p_\phi \phi + \int \sqrt{\frac{1}{2}mE + \frac{\beta}{2\xi} - \frac{ma(\xi)}{2\xi} = \frac{p_\phi^2}{4\xi^2}} d\xi + \\ \int \sqrt{\frac{1}{2}mE - \frac{\beta}{2\eta} - \frac{mb(\eta)}{2\eta} - \frac{p_\phi^2}{4\eta^2}} d\eta \end{aligned}$$

β can be determined from (155) and (156)

$$2\xi p_\xi^2 + ma(\xi) - mE\xi + \frac{p_\phi^2}{2\xi} = \beta \tag{171}$$

$$2\eta p_\eta^2 + mb(\eta) - mE\eta + \frac{p_\phi^2}{2\eta} = \beta \tag{172}$$

Subtracting

$$2\xi p_\xi^2 - 2\eta p_\eta^2 + ma(\xi) - mb(\eta) - mE\xi + mE\eta + \frac{p_\phi^2}{2\xi} - \frac{p_\phi^2}{2\eta} = 2\beta \tag{173}$$

Transforming from p_ξ and p_η to $p_\rho = \partial S/\partial \rho$ and $p_z = \partial S/\partial z$ we get

$$\beta = -m\left[\frac{\alpha z}{r} + \frac{p_\rho}{m}(zp_\rho - \rho p_z) + \frac{p_\phi^2}{m\rho^2}z\right] - \frac{1}{2}mF\rho^2 \tag{174}$$

Compare this with the conservation law discussed in formulas (94) through (95) from Lesson 4 (page 19).

$$\vec{v} \times \vec{M} + \alpha \frac{\vec{r}}{r} \tag{175}$$

which is a conserved quantity for a particle moving in the effective potential

$$U_{eff} = \frac{\alpha}{r} + \frac{M^2}{2mr^2} \tag{176}$$

for potentials $U = \alpha/r$

Let's calculate the total derivative of (175) respect to time:

$$\frac{d}{dt}\left(\vec{v} \times \vec{M} + \alpha \frac{\vec{r}}{r}\right) \tag{177}$$

we get

$$\vec{\dot{v}} \times \vec{M} + \alpha \frac{\vec{v}}{r} - \alpha(\vec{v} \cdot \vec{r}) \frac{\vec{r}}{r^3} \tag{178}$$

Since $M = m\vec{r} \times \vec{v}$ we have

$$m\vec{r}(\vec{v} \cdot \vec{\dot{v}}) - m\vec{v}(\vec{r} \cdot \vec{\dot{v}}) + \alpha \frac{\vec{v}}{r} - \alpha \frac{r}{r^3}(\vec{v} \cdot \vec{r}) \tag{179}$$

Using that $m\vec{\dot{v}} = \alpha \frac{\vec{r}}{r^3}$ we obtain

$$\frac{d}{dt}(\vec{v} \times \vec{M} + \alpha \frac{\vec{r}}{r}) = 0 \tag{180}$$

The direction is along the major axis from the focus to the perihelion and its magnitude is αe.

Example 2

Find the H-J equations of motion if the field is

$$U = \frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2} \tag{181}$$

This is he Coulomb field of two fixed charges at a distance 2σ apart.

Solution

In this case we can use elliptical coordinates. The field is

$$U = \frac{a(\xi) + b(\eta)}{\xi^{-}\eta^2} \tag{182}$$

with

$$a(\xi) = \frac{(\alpha_1 + \alpha_2)\xi}{\sigma}, \qquad a(\xi) = \frac{(\alpha_1 - \alpha_2)\eta}{\sigma} \tag{183}$$

The action is given by (165)

$$S = -Et + p_\phi \phi + \int \sqrt{2m\sigma^2 E + \frac{\beta - 2m\sigma^2 a(\xi)}{\xi^2 - 1} - \frac{p_\phi^2}{(\xi^2-1)^2}} d\xi + \int \sqrt{2m\sigma^2 E - \frac{\beta + 2m\sigma^2 b(\eta)}{1-\eta^2} - \frac{p_\phi^2}{(1-\eta^2)^2}} d\eta$$

with the conserved quantities given by

$$\beta = \sigma^2 \left(p^2 + \frac{p_\phi^2}{\rho^2} \right) - M^2 + 2m\sigma(\alpha_1 \cos\theta_1 + \alpha_2 \cos\theta_2) \tag{184}$$

$$M^2 = (\vec{r} \times \vec{p})^2 = p^2 z^2 + p_z^2 + r^2 \frac{p_\phi^2}{\rho^2} - 2z\rho p_z p_\rho \tag{185}$$

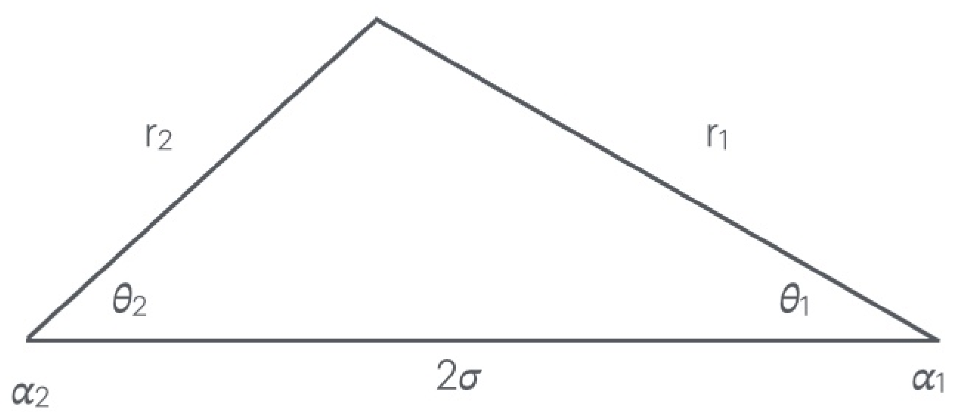

Figure 2: The field at a point of space r_1 and r_2 from charges located at α_1 and α_2 separated 2σ.

Adiabatic Invariants

Consider a mechanical system executing a finite motion in 1-dimension, characterized by some parameter λ which is related to the properties of the system.
Let's suppose that λ varies adiabatically (adiábatos in Greek: impassable). It means λ varies slowly.

$$T\frac{d\lambda}{dt} \ll \lambda \tag{186}$$

where T is the period of motion.

If λ is variable then the energy E is not conserved. If λ were constant the system would be closed (i..e periodic motion with constant energy E and fixed period $T(E)$. But if $\Delta\lambda$ is small $\Delta E/\Delta t = \dot{E}$ will also be small.
Then $\partial E/\partial t \propto \partial\lambda/\partial t$ and the E dependence on λ can be expressed as some constant combination of E and λ. This constant which remains as such throughout motion is called an adiabatic invariant.

Let $H(q, p, \lambda$ be the aHamiltonian of the system. Then

$$\frac{dE}{dt} = \frac{\partial H}{\partial t} = \frac{\partial H}{\partial \lambda}\frac{\partial \lambda}{\partial t} \tag{187}$$

$\frac{\partial H}{\partial \lambda}\frac{\partial \lambda}{\partial t}$ depends not only on λ but also on p and q which may be changing rapidly. It is appropriate to average over a period of motion

$$\overline{\frac{dE}{dt}} = \frac{d\lambda}{dt}\overline{\frac{\partial H}{\partial \lambda}} \tag{188}$$

and we will assume that λ varies slowly. $\overline{\frac{\partial H}{\partial \lambda}}$ is a function of q, p only, and

$$\overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T}\int_0^T \frac{\partial H}{\partial \lambda}dt \tag{189}$$

where

$$\dot{q} = \frac{\partial H}{\partial p} \tag{190}$$

So

$$dt = \frac{dq}{\frac{\partial H}{\partial p}}$$

and

$$T = \int_0^T dt = \oint \frac{dq}{\frac{\partial H}{\partial p}}$$

where $\oint$ refers to a cycle integration.
Then

$$\overline{\frac{dE}{dt}} = \frac{d\lambda}{dt} \oint \frac{\frac{\partial H}{\partial \lambda}}{\frac{\partial H}{\partial p}} dq \frac{1}{\oint \frac{dq}{\frac{\partial H}{\partial p}}} \tag{191}$$

The integrations are performed over paths of constant λ. Along such paths the hamiltonian has constant value E, and the momentum is a function of q and E and λ, so $p = p(q, E, \lambda)$ and if we differentiate respect to λ the equation $H(q, p\lambda) = E$ we get

$$\frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial \lambda} = 0 \rightarrow \frac{\frac{\partial H}{\partial \lambda}}{\frac{\partial H}{\partial p}} = -\frac{\partial p}{\partial \lambda}$$

which substituted in (191) gives

$$\overline{\frac{dE}{dt}} = \frac{d\lambda}{dt} \frac{\oint \frac{\partial p}{\partial \lambda} dq}{\oint \frac{\partial p}{\partial E} dq}$$

which is equivalent to

$$\oint \left(\frac{\partial p}{\partial E} \overline{\frac{dE}{dt}} + \frac{\partial p}{\partial \lambda} \frac{\partial \lambda}{\partial t} \right) dq = 0 \tag{191}$$

Notice that if we define

$$I \equiv \oint p \frac{dq}{2\pi} \tag{192}$$

we would get

$$\overline{\frac{dI}{dt}} = 0 \tag{193}$$

I is an adiabatic invariant (it remains constant when λ varies). If we calculate $\partial I/\partial E$ we remember that the period

$$T = \int_0^T dt = \oint \frac{dq}{\frac{\partial H}{\partial p}}$$

and using (192)

$$2\pi \frac{\partial I}{\partial E} = \oint \frac{\partial p}{\partial E} dq = T \tag{194}$$

which is equivalent to

$$\frac{\partial E}{\partial I} = \omega \tag{195}$$

where $\omega = 2\pi/T$ is the vibration frequency of the system.

It is relevant to notice that (192)

$$I \equiv \oint p \frac{dq}{2\pi}$$

has a geometric meaning: the phase space of a system undergoing periodic motion is a closed curve in (p, q). As the momentum is also bound it can be written as

$$I = \int \int dp dq \frac{1}{2\pi} \tag{196}$$

and this is precisely the area enclosed by the path followed by the system in phase space.

Example

The 1-D oscillator. It has

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

ω is the frequency of the oscillator. The equation of the phase space path is given by $H(p, q) = E$ Which is given in phase space by the ellipse in figure 3

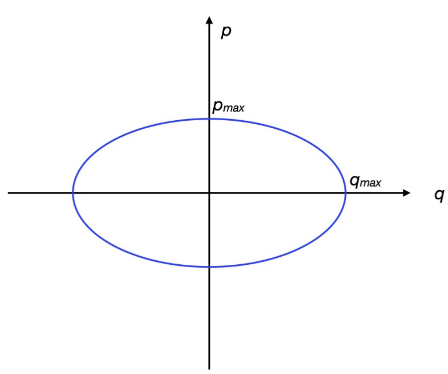

Figure 3: The path of a simple 1-D harmonic oscillator in phase space

We can see that the maximum values of p and q are obtained from

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 = E \tag{197}$$

when $\dot{q} = 0, \quad p = 0$ and q is maximum

$$q_{max} = \sqrt{\frac{2E}{m\omega^2}}$$

when $q = 0$ p is maximum

$$p_{max} = \sqrt{2mE}$$

and the area of the ellipse is

$$A = \pi q_{max} p_{max} = \frac{2\pi E}{\omega} \tag{198}$$

and then

$$A = \pi q_{max} p_{max} = \frac{2\pi E}{\omega} \tag{199}$$

and

$$I = \int\int dpdq \frac{1}{2\pi} = \frac{E}{\omega} \tag{200}$$

When the parameters of the oscillator the energy is proportional to the frequency.

Canonical variables

Let λ be constant, so that the system is closed. We carry out a canonical transformation for q and p using I as a new "momentum". The generating function is $S_0 = S_0(q, I)$

$$S_0(q, E, \lambda) = \int p(q, E, \lambda) dq \tag{201}$$

For a closed system $(\lambda = ct)$ I is only a function of the energy E. This implies we can think of $S_0(q, I, \lambda)$ and

$$\left(\frac{\partial S_0}{\partial q}\right)_E = \left(\frac{\partial S_0}{\partial q}\right)_I \tag{202}$$

for constant I. Then according to the formulas in (101)

$$p_i = \frac{\partial \Phi}{\partial q_i}, \qquad Q_i = \frac{\partial \Phi}{\partial P_i}, \qquad H' = H + \frac{\partial \Phi}{\partial t}$$

we get

$$p = \frac{\partial S_0(q, I, \lambda)}{\partial q} \tag{203}$$

but we also get

$$\mathcal{W} = \frac{\partial S_0(q, I, \lambda)}{\partial I} \tag{204}$$

The canonical variables I and $\mathcal{W}$ are called the action variable and the angle variable respectively. $S_0(q, I, \lambda)$ as a generating function does not depend on time. So the new Hamiltonian H' is just $E(I)$ as a function of the action variable.
Hamilton's equations in canonical variables are

$$\dot{I} = 0 \qquad \dot{\mathcal{W}} = \frac{dE(I)}{dI} \tag{205}$$

The first one just shows that E is constant. On the other hand the angle variable is a linear function of time

$$\frac{d\mathcal{W}}{dt} = \frac{dE(I)}{dI} \rightarrow \mathcal{W} = \frac{dE(I)}{dI} dt + constant \tag{206}$$

or

$$\mathcal{W} = \omega(I)t + constant \tag{207}$$

$\mathcal{W}$ is the phase of the oscillation. The action $S_0(q, I)$ is a many valued function of the coordinates. During each period the function increases by

$$\Delta S_0 = 2\pi I \tag{208}$$

We should remember that

$$S_0(q, E, \lambda) = \int p(q, E, \lambda) dq$$

and

$$I = \oint \frac{pdq}{2\pi}$$

During the same time the angle variable

$$\Delta\omega = \Delta\left(\frac{\partial S_0}{\partial I}\right) = \frac{\partial \Delta S_0}{\partial I} = 2\pi \tag{209}$$

On the other hand if we expressed q and p or any one-valued function $F(q,p)$ in terms of I and $\mathcal{W}$ they would remain unchanged when $\mathcal{W}$ increases by 2π (I constant). Any one function $F(q,p)$ when expressed in terms of I and $\mathcal{W}$ is a periodic function of in terms of $\mathcal{W}$ with period 2π.

Non closed systems

Canonical variables can also be used for a system that it is not closed, in which the parameter λ is time dependent.

As before

$$S_0(q,E,\lambda) = \int p(q,E,\lambda)dq$$
$$I = \oint \frac{pdq}{2\pi}$$

but this time we have $\lambda = \lambda(t)$.
The generating function is now an explicit function of time. The new Hamiltonian H' is different from the old one which was the energy $E(I)$.
Let's remember that if we transform from coordinates $p,q \to P,Q$ we have a generating function $F(q,Q) \to \Phi(q,P)$ and then,

$$p_i = \frac{\partial \Phi}{\partial q_i} \qquad Q_i = \frac{\partial \Phi}{\partial P_i} \qquad H' = H + \frac{\partial \Phi}{\partial t} \tag{210}$$

Then we have S_0 the generating function and

$$\begin{aligned} H' =& E(I,\lambda) + \frac{\partial S_0}{\partial t} \\ =& E(I,\lambda) + \left(\frac{\partial S_0}{\partial \lambda}\right)_{q,t} \frac{\partial \lambda}{\partial t} \\ =& E(I,\lambda) + \Lambda\dot{\lambda} \end{aligned} \tag{211}$$

where Λ should be written expressly as $\Lambda(I,\mathcal{W})$ after taking the time derivative.

Hamilton's equations for closed systems are

$$\dot{I} = 0 \qquad\qquad \dot{\mathcal{W}} = \frac{dE(I)}{dI}$$

and then in our case

$$\frac{\partial I}{\partial t} = -\frac{H'}{\mathcal{W}} = -\frac{\partial \Lambda}{\partial \mathcal{W}}\frac{\partial \lambda}{\partial t} \tag{212}$$

remember $\partial Q_i/\partial t = \partial H'/\partial \mathcal{W}$ and

$$\frac{\partial 5}{\partial t} = -\frac{\partial H'}{\partial I} = \frac{dE}{dI} + \frac{\partial \Lambda}{\partial I}\dot{\lambda} = \omega(I,\lambda) + \left(\frac{\partial \Lambda}{\partial I}\right)_{\mathcal{W}\lambda}\dot{\lambda} \tag{213}$$

$$\omega(\neq \mathcal{W}) = \left(\frac{\partial E}{\partial I}\right)_{\lambda}$$

is the oscillation frequency calculated as if λ were constant.

Example

Write the EoM in canonical variables for a harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(t)q^2 \tag{214}$$

$$q = \sqrt{\frac{2E}{m\omega^2}}\sin\omega(t) = \sqrt{\frac{2I}{m\omega}}\sin\omega(t) \tag{215}$$

$$p = \sqrt{2I\omega m}\cos\omega(t) \tag{216}$$

And

$$S_0 = \int pdq = \int p\left(\frac{\partial q}{\partial \mathcal{W}}\right)_{t,\omega} d\omega = 2I\int \cos^2\omega d\omega \tag{217}$$

$$\Lambda = \left(\frac{\partial S_0}{\partial \omega}\right)_{q,I} = \left(\frac{\partial S_0}{\partial \mathcal{W}}\right)_t \left(\frac{\partial \mathcal{W}}{\partial \omega}\right)_q = \frac{I}{2\omega}\sin\mathcal{W} \tag{218}$$

Then

$$\dot{I} = -I\left(\frac{\dot{\omega}}{2\omega}\right)\cos 2\mathcal{W} \tag{219}$$

$$\dot{\mathcal{W}} = \omega + \left(\frac{\dot{\omega}}{2\omega}\right)\sin 2\mathcal{W} \tag{220}$$