Classical Mechanics 2025
Lesson 7:

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1 Rigid Bodies

In this chapter we will be dealing with extended rigid bodies. How do we define then a body that is extended in volume and it is solid or rigid? We can think of a system of particles, a lot many of them, which keep their distances among them constant.
Of course solid mechanics could be treated within the framework of continuous media. Think of properties like elasticity, compressibility, and similar characteristics of extended bodies. But this is not course of mechanic of continuous media. So we will limit ourselves to a basic formulation where a treatment of a system of particles which retain their relative distances constant throughout motion.

Also in order to describe the motion of a rigid body we will use two coordinate systems. A "fixed" (inertial) system X, Y, Z and a second one attached to the body x_1, x_2, x_3 (see Figure 8).

The other thing to do to develop the theory of rigid bodies is to go from a sum of the particles to an integration: i.e. from $\sum m$ to ρdV, where ρ is the density and dV a differential of volume for the element we are considering.

The x_1, x_2, x_3 system is located at the center of mass of the rigid body we are studying. Figure 8) shows the relationship between the description of a point in the body from the two systems of reference: $\vec{R}$ is the position vector of the center of mass of the body in the X, Y, Z inertial system of reference. The orientation of the axes x_1, x_2, x_3 respect to the x_1, x_2, x_3 system is given by three independent angles. These with the 3 components of $\vec{R}$ gives a total of six coordinates. A rigid body is a mechanical system with six degrees of freedom.

We can now consider an arbitrary infinitesimal displacement of our rigid body under study. We can describe it as a sum of two components:
1) A translation of the body as if it were a point like particle with its entire mass positioned at its center of mass, and,
2) an infinitesimal rotation of the body itself around its center of mass.

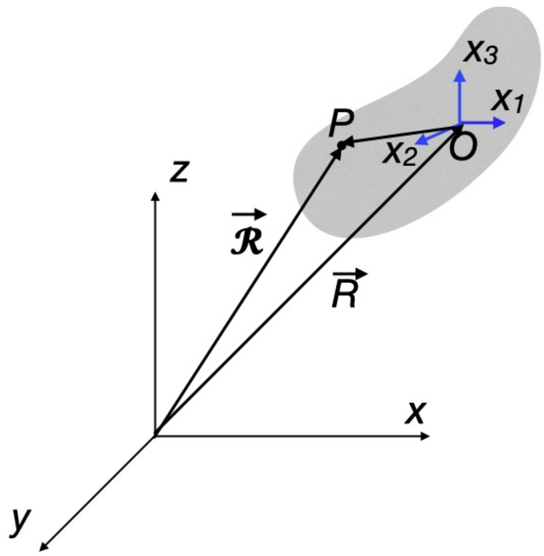

Figure 1: The systems of coordinates use to describe the motion of a rigid body

If $\vec{r}$ is the vector of any point P located in the rigid body from the co-moving system of reference, $\vec{\mathfrak{R}}$ the vector of the same point from the external system of reference X, Y, Z, then an infinitesimal displacement $d\vec{\mathfrak{R}}$ of P consists of a displacement $d\vec{R}$ of the center of mass plus a displacement $d\vec{\phi} \times \vec{r}$ relative to the center of mass. This latter one is the result of a rotation through an infinitesimal angle $d\vec{\phi}$ around and axis passing through the center of mass.

$$d\vec{\mathfrak{R}} = d\vec{R} + d\vec{\phi} \times \vec{r} \tag{1}$$

If we take this equation and consider it in a differential of time dt we will have

$$\frac{d\vec{\mathfrak{R}}}{dt} = \vec{v}; \qquad \frac{d\vec{R}}{dt} = \vec{V}; \qquad \frac{d\vec{\phi}}{dt} = \vec{\Omega}, \tag{2}$$

and then

$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}, \tag{3}$$

$\vec{V}$ is the velocity of the body center of mass and also the translational velocity of the body itself. $\vec{\Omega}$ is the angular velocity of the body. Its direction, that of $d\vec{\phi}$ is along the axis of rotation.

The velocity of any point of the rigid body can be expressed as a sum of its translational velocity $\vec{V}$ and its angular rotation.

2 Uniqueness of Angular Velocity

Let's assume that the $\{x_1, x_2, x_3\}$ coordinate system is not at the center of mass (see Figure 2). The position of the origin is of this system, O', is given by the vector $\vec{a}$ from O. Let's call the velocity of O' $\vec{V}'$ and the angular velocity as measured from O', $\vec{\Omega}'$.

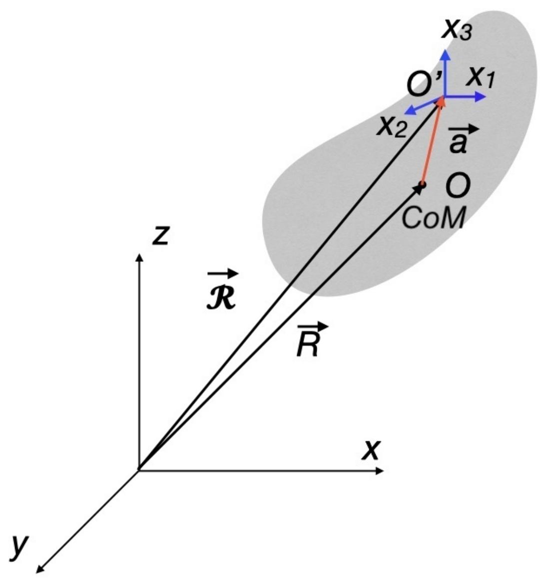

Figure 2: The systems of coordinates $\{x_1, x_2, x_3\}$ away from the Center of Mass of the body.

Pick a point on the body P identified from respect to O' by a vector $\vec{r}'$ such that

$$\vec{r} = \vec{r}' + \vec{a} \tag{4}$$

so we get

$$\begin{aligned}\vec{v} =& \vec{V} + \vec{\Omega} \times \vec{r} = \vec{V} + \vec{\Omega} \times (\vec{r}' + \vec{a}) = \\ =& \vec{V} + \vec{\Omega} \times \vec{r}' + \vec{\Omega} \times \vec{a}\end{aligned} \tag{5}$$

But at the same time we have in O' that

$$\vec{v} = \vec{V}' + \vec{\Omega}' \times \vec{r}' \tag{6}$$

But eq (5) is compatible with (6) only if

$$\vec{V}' = \vec{V} + \vec{\Omega} \times \vec{a}, \qquad \vec{\Omega}' = \vec{\Omega} \tag{7}$$

This is a very important result: at any instant the angular velocity of rotation of a system of coordinates fixed to the body is independent of the particular system chosen. All rigid bodies rotate with one unique angular velocity Ω which is the same regardless of the coordinate system inside the body chosen to describe it. Any $\vec{\Omega}'$ calculated from a system of coordinates not coincident with the center of mass of the body will be parallel and of the same magnitude to the one calculated from the center of mass system.

If at any point it happens that $\vec{V}$ is perpendicular to $\vec{\Omega}$ then any $\vec{V}'$ will also be perpendicular to $\vec{\Omega}'$. From equation (3) $\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$, we can see that the velocities $\vec{v}$ for all points in the rigid body are perpendicular to $\vec{\Omega}$.

We can then choose a system with origin O' such that $\vec{V}'$ is 0. In that case the motion of the body at the instant considered is a pure rotation around an axis through O'. This is called the instantaneous axis of rotation. In that case, of course, the origin of the moving rigid body is at the center of mass of it.

3 The Inertia Tensor

We can calculate the kinetic energy of the extended body, considering it as a system composed of a discrete number of particles:

$$\begin{aligned} T =& \sum \frac{1}{2} m v^2 = \sum \frac{1}{2} m \left(\vec{V} + \vec{\Omega} \times \vec{r}\right)^2 \\ =& \sum \frac{1}{2} m V^2 + \sum \frac{1}{2} m \left(\vec{\Omega} \times \vec{r}\right)^2 + \sum m \vec{V} \cdot \vec{\Omega} \times \vec{r} \end{aligned} \tag{8}$$

With $\sum m = \mu$ and taking into account that

$$\sum m \vec{V} \cdot \vec{\Omega} \times \vec{r} = \sum m \vec{r} \cdot \vec{V} \times \vec{\Omega} = \vec{V} \times \vec{\Omega} \cdot \sum m \vec{r} = 0 \tag{9}$$

where $\sum m \vec{r} = 0$ because it is the radius vector of the center of mass and we have picked our coordinate system centered there.

Also

$$\left(\vec{\Omega} \times \vec{r}\right)^2 = \left(\vec{\Omega} \times \vec{r}\right) \cdot \left(\vec{\Omega} \times \vec{r}\right) = |\vec{\Omega} \times \vec{r}|^2 = \Omega^2 r^2 \sin^2\theta \tag{10}$$

where θ is the angle between $\vec{\Omega}$ and $\vec{r}$ and of course we can write as $\sin^2\theta = 1 - \cos^2\theta$ which when used in (10) gives

$$|\vec{\Omega} \times \vec{r}|^2 = \Omega^2 r^2 - \Omega^2 r^2 \cos^2\theta = \Omega^2 r^2 - \left(\vec{\Omega} \cdot \vec{r}\right)^2 \tag{11}$$

The kinetic energy then can be expressed

$$T = \frac{1}{2}\mu V^2 + \frac{1}{2}\sum m \left(\Omega^2 r^2 - \left(\vec{\Omega} \cdot \vec{r}\right)^2\right) \tag{12}$$

where $\frac{1}{2}\mu V^2$ is the translational kinetic energy. The second term is kinetic energy of only rotation with angular velocity $\vec{\Omega}$ about an axis passing through the center of mass of the body in question. We can write this term in tensor form:

$$\begin{aligned} T_{rot} = \frac{1}{2}\sum m \left({\Omega_i}^2 {x_i}^2 - \Omega_i x_i \Omega_k x_k\right) =& \frac{1}{2}\sum m \left(\Omega_i \Omega_k \delta_{ik} x_l^2 - \Omega_i \Omega_k x_i x_k\right) \\ =& \Omega_i \Omega_k \sum m \left(x_l^2 \delta_{ik} - x_i x_k\right) \end{aligned} \tag{13}$$

where $\Omega_i = \delta_{ik}\Omega_k$ and in 3 dimensions

$$\delta_{ik} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{14}$$

I_{ik} is called the inertia tensor:

$$I_{ik} = \sum m \left(x_l^2 \delta_{ik} - x_i x_k\right) \tag{15}$$

Then, in general the kinetic energy of rigid body undergoing translation and rotation can be written as

$$T = \frac{1}{2}\mu V^2 + \frac{1}{2} I_{ik} \Omega_i \Omega_k \tag{16}$$

and the complete Lagrangian function is

$$L = \frac{1}{2}\mu V^2 + \frac{1}{2} I_{ik} \Omega_i \Omega_k - U \tag{17}$$

and

$$I_{ik} = I_{ki} \tag{18}$$

i.e. the inertia tensor is symmetric.

$$I_{ik} = \begin{pmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum myx & \sum m(x^2+z^2) & -\sum myz \\ -\sum mzx & -\sum mzy & \sum m(x^2+y^2) \end{pmatrix} \tag{19}$$

For a continuous body we can generalize the definition formula (15) to

$$I_{ik} = \int \rho \left(x_l^2 \delta_{ik} - x_i x_k \right) dV \tag{20}$$

where ρ is the density of body in question and the integral is performed over the entire volume occupied by it.

The inertia tensor is a symmetric tensor of rank 2. It can be reduced to diagonal form by an appro-

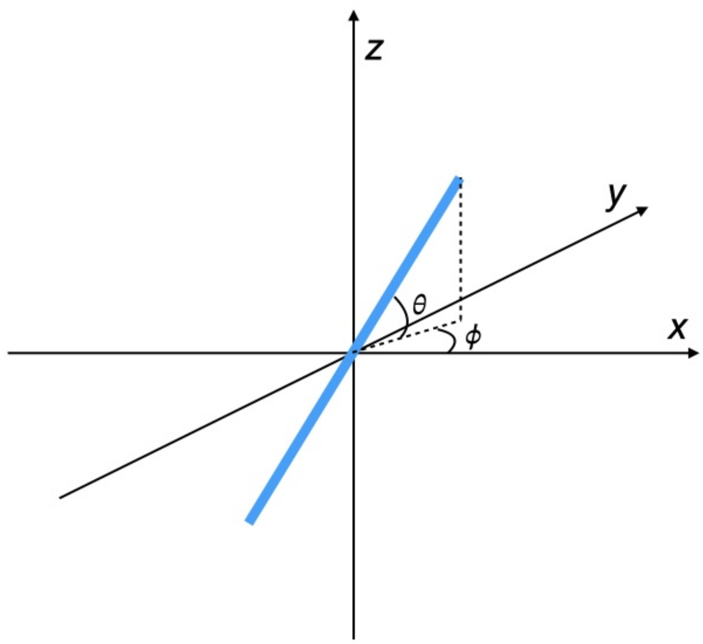

Figure 3: A rigid 2-D rotator

priate choice of coordinates. When the tensor is diagonalized the axis will result aligned with the principal directions determined by the new coordinate system. In that case the axis of symmetry of the body in new system are called the principal moments of inertia, I_1, I_2, I_3.

In a case like this

$$T_{rot} = \frac{1}{2} \left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 \right) \tag{21}$$

None of these principal moments can exceed the sum of the other two. For example we can see that

$$I_1 + I_2 = \sum m \left(x_1^2 + x_3^2 + x_2^2 + x_3^2 \right) \geqslant \sum m \left(x_1^2 + x_2^2 \right) = I_3 \tag{22}$$

A body with $I_1 \neq I_2$, $I_1 \neq I_3$ and $I_2 \neq I_3$ is called an asymmetrical top. If any two of the principal moments of inertia are equal it is called a symmetrical top, and if all moments are equal it is called a spherical top.

Examples

Example 1:

a) Using equation (20) write the inertia tensor for the 2- D rigid rotator pictured in Figure 3. Notice that you will obtain a matrix with all the entries different from zero and in terms of $\cos$ and $\sin$ of the angles involved.

b) align the rotator in such a way that it can rotate with its axis being any of the coordinate axes (i.e. x, y or z) and show that the moment inertia around that axis is $ml/12$ where m is the mass of the rotator and l is its length.

c) If

$$M = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{23}$$

and

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{24}$$

Then $MNIN^{-1}M^{-1} = I'$ where I is the rotational inertia obtain in a) using formula (2)) and I' is a diagonal matrix.

$$I' = \begin{pmatrix} ml^2/12 & 0 & 0 \\ 0 & ml^2/12 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{25}$$

In the case that the axis of rotation doesn't pass through the center of mass we would have

$$\vec{r} = \vec{r}' + \vec{a} \tag{26}$$

equivalent to

$$x_i = x_i' + a_i \tag{27}$$

So we get

$$I'_{ik} = \sum m \left(x'^2_l \delta_{ik} - x'_i x'_k \right) \tag{28}$$

which when we use (27) gives

$$I'_{ik} = I_{ik} + \mu \left(a^2 \delta_{ik} - a_i a_k \right) \tag{29}$$

Example 2

Determine the principal moments of inertia for the following molecules

a) A molecule made of collinear atoms.

Figure 4: a linear molecule of n atoms

$$I_1 = I_2 = \frac{1}{\mu} \sum_{a \neq b} m_a m_b l_{ab}^2 \tag{30}$$

and $I_3 = 0$ where $\mu = \sum_a m_a$. In the particular case of a diatomic molecule

$$I_1 = I_2 = \frac{m_1 m_2}{m_1 + m_2} l^2 = m l^2 \tag{31}$$

where m is the reduced mass.

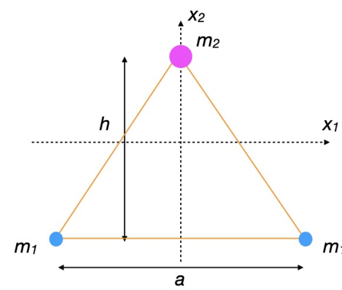

Figure 5: a triatomic molecule

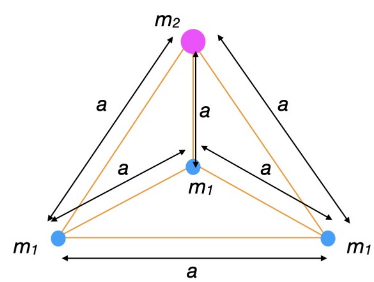

Figure 6: a tetratomic molecule

b) a triatomic molecule forming an isosceles triangle (Fig 5) .

The center of mass is at $x_2 = m_2 h/\mu$, where $\mu = 2m_1 + m_2$ the total mass of the molecule. Then $I_1 = 2m_1 m_2 h^2/\mu$, $I_2 = (1/2)m_1 a^2$ and $I_3 = I_1 + I_2$.

c) a tetratomic molecule forming an isosceles triangle (Fig 6) .

The centre of mass is on the axis of symmetry of the tetrahedron at a distance $x_2 = m_2 h/\mu$ where h is the height of the tetrahedron.
$I_1 = I_2 = 3m_1 m_2 h^2/\mu + \frac{1}{2}m_1 a^2$ and $I_3 = m_1 a^2$.
If $m_1 = m_2$ then $h = \sqrt{2/3}a$ and $I_1 = I_2 = \frac{3m_1^2}{4m_1}\frac{2}{3}a^2 + \frac{1}{2}m_1 a^2 = \frac{1}{2}m_1 a^2 + \frac{1}{2}m_1 a^2 = I_3$.

4 Angular Momentum of a Rigid Body

The value of the angular momentum of a body depends on the point respect to which it is defined. The most convenient one is the center of mass of the body. We will call it $\vec{M}$. And when we use it we will be considering the intrinsic angular momentum resulting from its motion relative to the Center of Mass.

In $\vec{M} = \sum m\vec{r} \times \vec{v}$ we will replace $\vec{v} = \vec{\Omega} \times \vec{r}$ to obtain

$$\vec{M} = \sum m\vec{r} \times \vec{\Omega} \times \vec{r} = \sum \left(r^2\vec{\Omega} - \vec{r}(\vec{r} \cdot \vec{\Omega}) \right)$$

which in tensorial notation is

$$\begin{aligned} M_i =& \sum m(x_l^2 \Omega_i - x_i x_k) \\ =& \Omega_k \sum m(x_l^2 \delta_{ik} - x_i x_k) \end{aligned} \tag{32}$$

which using the definition of the inertia tensor becomes

$$M_i = I_{ik}\Omega_k \tag{33}$$

And if I_{ik} is in diagonal form

$$M_1 = I_1\Omega_1 \qquad M_2 = I_2\Omega_2 \qquad M_3 = I_3\Omega_3 \tag{34}$$

For a spherical top

$$\vec{M} = \mathrm{I}\vec{\Omega} \tag{35}$$

Conservation of angular momentum means that when we have a free rotation (no translation) it will happen in a plane about an axis perpendicular to it. This is easy to visualize for a spherical body or a rotator.

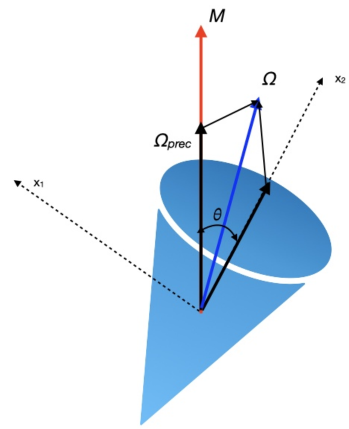

Figure 7: a symmetrical top

5 The symmetrical top

In the case of a symmetrical top we will take advantage that the principal axes of inertia x_1, x_2 (the axes perpendicular to x_3, the axis of symmetry of the top) can be chosen arbitrarily.
We will take then the x_2 axis perpendicular to the plane containing $\vec{M}$ and the axis x_3.
Then $M_2 = 0$ and $\Omega_2 = 0$. This implies that $\vec{M}$ and $\vec{\Omega}$ are instantaneously on one plane. Which itself implies that $\vec{v} = \vec{\Omega} \times \vec{r}$ for every point on the axis of the top is instantaneously perpendicular to that plane.

The axis of the top rotates uniformly around the direction of $\vec{M}$, describing a circular cone: this phenomenon is called a regular precession of the top.

Of course, additionally the top rotates uniformly around its axis. The angular velocity of the top is just

$$\Omega_3 = \frac{M_3}{I_3} = \left(\frac{M}{I_3}\right)\cos\theta \tag{36}$$

We can see that

$$\Omega_{prec}\sin\theta = \Omega_1 \tag{37}$$

and due to the fact that

$$\Omega_1 = \frac{M_1}{I_1} = \left(\frac{M}{I_1}\right)\sin\theta \tag{38}$$

we have

$$\Omega_{prec} = \frac{M}{I_1} \tag{39}$$

6 The equations of motion of a rigid body

A rigid body has 6 degrees of freedom, in general. So we have 6 equations of motion. The total momentum of the body is

$$\vec{P} = \sum \vec{p} = \mu \vec{V} \tag{40}$$

The total force acting on it is

$$\vec{F} = \sum \vec{f} \tag{41}$$

and

$$\frac{d\vec{P}}{dt} = \vec{F} \tag{42}$$

with, in general

$$\vec{F} = -\frac{\partial U}{\partial \vec{R}} \tag{43}$$

where differentiation takes place with respect to the body's center of mass.

When the body undergoes a translation δR there is an associated change in potential energy

$$\delta U = \sum \frac{\partial U}{\partial \vec{r}} \cdot \delta \vec{r} = \delta \vec{R} \cdot \sum \frac{\partial U}{\partial \vec{r}} = -\delta \vec{R} \cdot \sum \vec{f} = -\vec{F} \cdot \delta \vec{R} \tag{44}$$

In the language of the Lagrangian formalism

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{V}}\right) = \frac{\partial L}{\partial \vec{R}} \tag{45}$$

for the Lagrangian

$$L = \frac{1}{2}\mu \vec{V}^2 + \frac{1}{2} I_{ik} \Omega_i \Omega_k - U \tag{46}$$

and the Euler Lagrange equations are

$$\frac{\partial L}{\partial \vec{V}} = \mu \vec{V} = \vec{P}, \qquad \frac{\partial L}{\partial \vec{R}} = -\frac{\partial U}{\partial \vec{R}} = \vec{F} \tag{47}$$

To write the equation of motion for the momentum we choose a fixed inertial frame of reference so that the center of mass is at rest in the instant considered.

$$\frac{d\vec{M}}{dt} = \frac{d}{dt}\left(\sum \vec{r} \times \vec{p}\right) = \sum \dot{\vec{r}} \times \vec{p} + \sum \vec{r} \times \dot{\vec{p}} \tag{48}$$

Our choice of the frame of reference where $\vec{V} = 0$ with $\vec{V}$the velocity of translation of the center of mass of the body, implies that at the same time $\dot{r} = \vec{v}$, and since $\vec{v}$ and$\vec{p} = m\vec{v}$ are clearly parallel, $\dot{\vec{r}} \times \vec{p} = 0$ and replacing $\vec{p}$ for the force

$$\frac{d\vec{M}}{dt} = \vec{K}, \qquad \vec{K} = \sum \vec{r} \times \vec{f} \tag{49}$$

$\vec{M}$ has been defined as the angular momentum respect to the center of mass (intrinsic angular momentum). A such it will remain the same when calculated from another inertial reference frame by Galilean Relativity. The vector $\vec{r} \times \vec{f}$ is called the momentum of the force and $\vec{K}$ is called the total torque. It includes only the external forces.

if we move the origin such that the position vectors change from $\vec{r}$ to $\vec{r'}$

$$\vec{r'} = \vec{r} - \vec{a} \tag{50}$$

then the torque

$$\vec{K} = \sum \vec{r} \times \vec{f} = \sum \vec{r'} \times \vec{f} + \sum \vec{a} \times \vec{f} \tag{51}$$

and

$$\vec{K} = \vec{K}' + \vec{a} \times \vec{F} \tag{52}$$

Equation (49) can also be appreciated as a Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \vec{\Omega}} = \frac{\partial L}{\partial \vec{\phi}} \tag{53}$$

for a rotational coordinate ϕ_i. And

$$\frac{\partial L}{\partial \Omega_i} = I_{ik}\Omega_k = M_i \tag{54}$$

Then

$$\delta U = -\sum \vec{f} \cdot \delta\vec{r} = -\sum \vec{f} \cdot \delta\vec{\phi} \times \vec{r} = -\delta\vec{\phi} \cdot \sum \vec{r} \times \vec{f} = -\vec{K} \cdot \delta\vec{\phi} \tag{55}$$

from where we can easily see that

$$\vec{K} = -\frac{\partial U}{\partial \vec{\phi}} \tag{56}$$

so that

$$\frac{\partial L}{\partial \vec{\phi}} = -\frac{\partial U}{\partial \vec{\phi}} = \vec{K} \tag{57}$$

If $\vec{F}$ and $\vec{K}$ are perpendicular we can always find $\vec{a}$ such that $\vec{K}' = 0$ in the following equation

$$\vec{K} = \vec{K}' + \vec{a} \tag{58}$$

The choice, of course, it is not unique. When $\vec{K}$ is perpendicular to $\vec{F}$ the effect of all the applied forces can be reduced to that of a single force acting along this line.

Example: a uniform field of force in which the force on a particle is $\vec{f} = e\vec{E}$. Then $\vec{F} = \vec{E}\sum \vec{e}$ and

$$\vec{K} = \sum e\vec{r} \times \vec{E}. \tag{59}$$

Assuming $\sum e \neq 0$ we can define a radius vector $\vec{r}_0$ such

$$\vec{r}_0 = \frac{\sum e\vec{r}}{\sum e}. \tag{60}$$

and then the torque is then simply

$$\vec{K} = \vec{r}_0 \times \vec{F}. \tag{61}$$

in a uniform field, the effect of the field reduces to the action of a single force $\vec{F}$ applied at the point whose radius vector is $\vec{r}_0$. The position of this field depends on the properties of the body (i.e. density and geometry). In a gravitational field such point is called the center of mass.

7 Euler angles

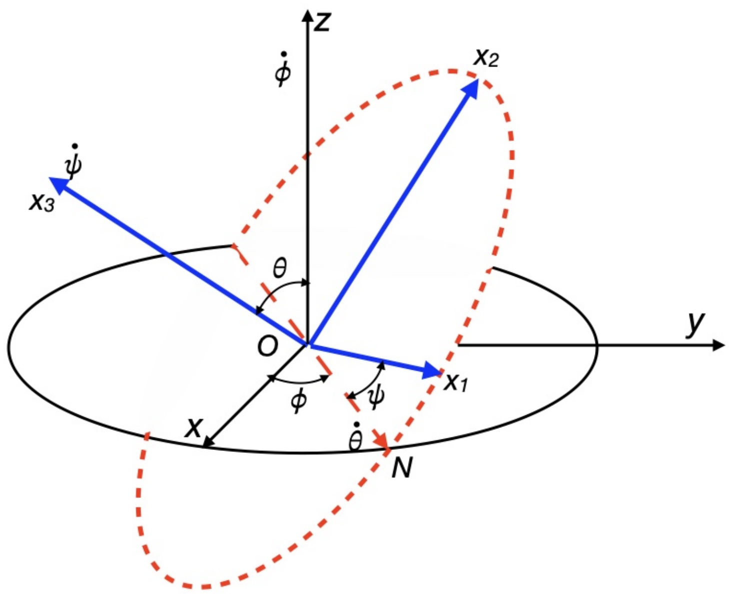

Figure 8: Euler angles

The six degrees of freedom of a body can be better described by the three coordinates labeling the position of its center of mass and 3 angles which determine the orientation of the axes x_1, x_2 and x_3 in the moving system of coordinates relative to the fixed system X, Y, Z. These angles are called Euler angles.

The origins of our systems x_1, x_2, x_3 and X, Y, Z coincide. The moving x_1x_2-plane intersects the fixed XY-plane defining a line called the lines of nodes ON in Figure 8. This line is by construction perpendicular to both the Z-axis and the x_3-axis. We take the positive direction of this line of nodes as that of the vector defined by $\vec{z} \times \vec{x_3}$ where these vectors are unit vectors along their respective axes.

Then we take as the angle defining the position of the axes x_1, x_2, x_3 relative to the X, Y, Z the angle θ between the Z and x_3 axes, the angle ϕ between the X-axis and the line ON and, the angle ψ between the x_1-axis and the line ON. The angles ϕ and ψ are measured around the Z and x_3 axes respectively in the direction given by the right hand rule. The angle θ takes values from 0 to π, and ψ from 0 to 2π.

We can now express the components of the angular velocity vector $\vec{\Omega}$ along the moving axis x_1, x_2, x_3 in terms of the Eulerian angles and their derivatives.

To do this we need to find the components along those axes of the angular velocities $\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$:
i) $\vec{\dot{\psi}}$ is in the direction of the x_3-axis.
ii) $\vec{\dot{\phi}}$ is in the direction of the Z-axis.
iii) $\vec{\dot{\theta}}$ is in the direction of $\vec{ON}$.

The components of these angular velocities along the x_1, x_2, x_3-axes are
i)

$\dot{\theta}_1 = \dot{\theta}\cos\psi$
$\dot{\theta}_2 = \dot{\theta}\sin\psi$

ii)

$\dot{\phi}_3 = \dot{\phi}\cos\theta$
$\dot{\phi}_{12} = \dot{\phi}\sin\theta$
with
$\dot{\phi}_1 = \dot{\phi}\sin\theta\sin\psi$ and
$\dot{\phi}_2 = \dot{\phi}\sin\theta\cos\psi$

and ψ entirely along the x_3-axis. This would give for $\vec{\Omega}$

$$\begin{aligned}\Omega_1 =&\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \Omega_2 =&\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \Omega_3 =&\dot{\phi}\cos\theta + \dot{\psi}\end{aligned} \tag{62}$$

If we calculate the principal moments of inertia of the body along the x_1, x_2, x_3-axes we will get for the rotational energy in the already obtained formula (21) copied below

$$T_{rot} = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right). \qquad (21)$$

that in the case of a symmetrical top where $I_1 = I_2 \neq I_3$

$$T_{rot} = \frac{1}{2}I_1\left(\vec{\dot{\phi}}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 \tag{63}$$

This can be further simplified for a symmetrical top using that $\{x_1, x_2\}$ are arbitrary. We can take x_1 along the line of nodes ON. Then $\psi = 0$ and we obtain

$$\Omega_1 = \dot{\theta}, \quad \Omega_2 = \dot{\phi}\sin\theta, \quad \Omega_3 = \dot{\phi}\cos\theta + \dot{\psi} \tag{64}$$

Let's determine the free motion of a symmetric top. We take the Z-axis in the direction of $\vec{M}$, x_3 along the direction of symmetry of the top. x_1 by construction is on the line of nodes ON.
Then

$$\begin{aligned} M_1 =& I_1\Omega_1 = I_1\dot{\theta} \\ M_2 =& I_2\Omega_2 = I_1\dot{\phi}\sin\theta \\ M_3 =& I_3\Omega_3 = I_3(\dot{\phi}\cos\theta + \dot{\psi}) \end{aligned} \tag{65}$$

Since x_1 is perpendicular to Z $\theta = 0$ and $\dot{\theta} = 0$ we get

$$\begin{aligned} M_1 =& 0, \\ M_2 =& M\sin\theta \\ M_3 =& \cos\theta, \end{aligned} \tag{66}$$

and

$$\dot{\theta} = 0, \qquad I_1\dot{\phi} = M, \qquad I_3(\dot{\phi}\cos\theta + \dot{\psi}) = M\cos\theta \tag{67}$$

where θ, the angle between the axis of the top and the direction of $\vec{M}$ is constant.
The velocity of precession is given by second equation in (67)

$$\dot{\phi} = \frac{M}{I_1} \tag{68}$$

and the angular velocity with which the top rotates around its axis is

$$\Omega_3 = \frac{M}{I_3}\cos\theta \tag{69}$$

8 Euler's equations

The equations of motion in section 6 are obtained from a fixed system of coordinates.
$\frac{d\vec{P}}{dt}$ and $\frac{d\vec{M}}{dt}$ are calculated respect to it.
But what we just saw is that the simplest relations between the components of the rotational angular momentum of a rigid body $\vec{M}$ and the components of the angular velocity can be obtained take the simplest form in the moving coordinate system of axis $x1, x_2, x_3$ attached to center of mass of the body.

Let's look for the equations transforming the equations of motion to coordinates $x1, x_2, x_3$.
The time derivative of a vector $\vec{A}$ respect to the fixed system of coordinates (assuming it is not changing in the moving system) is $d\vec{A}/dt = \vec{\Omega} \times \vec{A}$. i.e. $\vec{A}$ changes only due to the rotation of the body. But in general would be

$$\frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + \vec{\Omega} \times \vec{A} \tag{70}$$

where d'/dt measure the rate of change respect to the moving system. using this we can write (43) and (49)

$$\frac{d'\vec{P}}{dt} + \vec{\Omega} \times \vec{P} = \vec{F} \qquad \frac{d'\vec{M}}{dt} + \vec{\Omega} \times \vec{M} = \vec{K} \tag{71}$$

We are taking the components of the equation above respect to the moving system x_1, x_2, x_3 like $\left(\frac{d'\vec{P}}{dt}\right)_1 = \frac{dP_1}{dt}$ etc... So replacing $\vec{P}$ by $\mu\vec{V}$ we get

$$\begin{aligned} \mu\left(\frac{dV_1}{dt} + \Omega_2 V_3 - \Omega_3 V_2\right) &= F_1, \\ \mu\left(\frac{dV_2}{dt} + \Omega_3 V_1 - \Omega_1 V_3\right) &= F_2 \\ \mu\left(\frac{dV_3}{dt} + \Omega_1 V_2 - \Omega_2 V_1\right) &= F_3 \end{aligned} \tag{72}$$

If the axes x_1, x_2, x_3 are the principal axes of inertia we can put $M_1 = I_1\Omega_1$ and get

$$\begin{aligned} I_1\frac{d\Omega_1}{dt} + (I_3 - I_2)\Omega_2\Omega_3 &= K_1, \\ I_2\frac{d\Omega_2}{dt} + (I_1 - I_3)\Omega_3\Omega_1 &= K_2 \\ I_3\frac{d\Omega_3}{dt} + (I_2 - I_1)\Omega_1\Omega_2 &= K_3 \end{aligned} \tag{73}$$

In a free rotation $\vec{K} = 0$.

Example

For a symmetrical top, $I_1 = I_2$ and from the third equation in (73) we get Ω_3 is a constant. The first two equations then can be written

$$\dot{\Omega}_1 = -\,\omega\Omega_2 \tag{74}$$

$$\dot{\Omega}_2 = \quad \omega\Omega_1 \tag{75}$$

where

$$\omega = \frac{\Omega_3(I_3 - I_1)}{I_1} \tag{76}$$

Multiplying (75) by i we can group both (74) and (75) as follows

$$\frac{d\left(\Omega_1 + i\Omega_2\right)}{dt} = i\omega\left(\Omega_1 + i\Omega_2\right) \tag{77}$$

differential equation which has as a solution

$$\Omega_1 + i\Omega_2 = A\exp(i\omega t). \tag{78}$$

which gives the real values

$$\Omega_1 = A\cos(\omega t), \tag{79}$$
$$\Omega_2 = A\sin(\omega t). \tag{80}$$

with A a real number. this shows that the component of the angular velocity perpendicular to the axis of the top (x_3) rotates with angular velocity ω with constant magnitude $A^2 = \Omega_1^2 + \Omega_2^2$ where

$$\Omega_1^2 = A^2\cos^2(\omega t), \tag{81}$$
$$\Omega_2^2 = A^2\sin^2(\omega t). \tag{82}$$

Since Ω_3 is also constant $\vec{\Omega}$ rotates uniformly with constant velocity ω around the axis x_3. $\vec{M}$ has a particular relationship with $\vec{\Omega}$ given by $M_1 = I_1\Omega_1$, $M_2 = I_2\Omega_2$ and $M_3 = I_3\Omega_3$ so the angular momentum $\vec{M}$ executes a motion similar to the axis of the top. In terms of the Eulerian angles the angular velocity of $\vec{M}$ about the x_3 axis is the same as $-\dot{\psi}$. From equation (67) in the section Euler angles

$$\dot{\theta} = 0, \qquad I_1\dot{\phi} = M, \qquad I_3(\dot{\phi}\cos\theta + \dot{\psi}) = M\cos\theta \quad (67)$$

we get

$$\dot{\psi} = \frac{M\cos\theta}{I_3} - \dot{\phi}\cos\theta = M\cos\theta\left(\frac{1}{I_3} - \frac{1}{I_1}\right). \tag{83}$$

or using (69)

$$-\dot{\psi} = \frac{\Omega_3(I_3 - I_1)}{I_1} \tag{84}$$