Classical Mechanics 2023
Lesson 3: Integration of the equations of motion

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Putting the integrals of motion to work

The most general Lagrangian of a 1-dimensional system can be written

$$L = \frac{1}{2}a(q)\dot{q}^2 - U(q) \tag{1}$$

where $a(q)$ is a function of the coordinates: think as an example of

$$a(r, \theta, \phi) = m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta)$$

where q_i are r, θ, ϕ. Of course in this case the Lagrangian in 3-D would be

$$L = \frac{1}{2}\sum_{i=1}^{3} a_i(q_i)\dot{q_i}^2 - U(q_i) \tag{2}$$

Of course in cartesian system in 1-D we have

$$L = \frac{1}{2}m\dot{x}^2 - U(x) \tag{3}$$

The conservation theorems provide a very useful framework for integrating a system as (3). From the theorem of conservation of energy we know:

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x) = E \tag{4}$$

If the energy E is constant we can write

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = E - U(x) \tag{5}$$

From where we obtain

$$\sqrt{\frac{1}{2}m}\frac{dx}{\sqrt{E - U(x)}} = dt \tag{6}$$

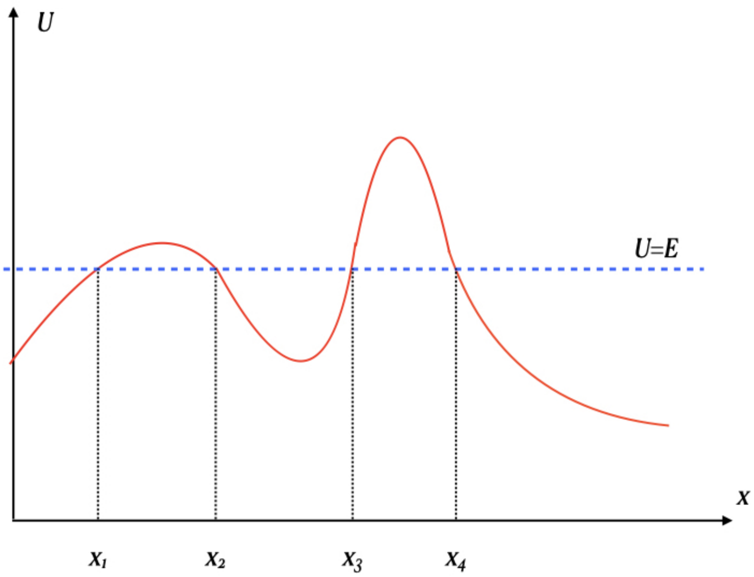

Figure 1: A generic potential curve with one particular value of the energy which remains constant

which we can solve:

$$t = \sqrt{\frac{1}{2}m} \int \frac{dx}{\sqrt{E - U(x)}} + constant \tag{7}$$

Since the kinetic energy is always positive the total energy is always greater than the potential energy. Let's take a look at Fig 1: At $U = E$ the kinetic energy is 0.
If the particle starts at a value of x such $x < x_1$ the particle is bound to move between 0 and x_1. It can not move between x_1 and x_2 or x_3 and x_4 because it would imply a negative kinetic energy. Those values of the potential present a potential barrier for the system. If the particle starts with a value $x_2 < x < x_3$ the motion will be finite, bound and oscillatory between x_2 and x_3. For a value of $x > x_4$ the motion will be unbound and the particle will escape to infinity with velocity $v = \sqrt{2E/m}$.

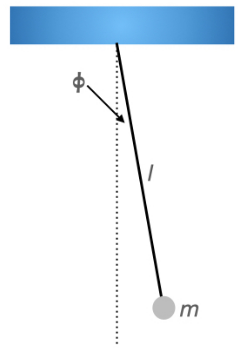

Figure 2: A pendulum of mass m and length l

In the case of oscillatory motion we can calculate the period. Using time reversibility (time symmetry) the period will be twice the time from x_1 to x_2:

$$T(E) = \sqrt{2m} \int_{x_2(E)}^{x_3(E)} \frac{dx}{\sqrt{E - U(x)}} \tag{8}$$

Notice that x_2 and $x_2 3$ are the roots of $E - U(x) = 0$. We would of course need to know the value of E.

Example

Find the period of oscillations of a simple pendulum of length l and mass m in a gravitational field (See Fig 2).

$$E = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl \cos\phi \tag{9}$$

At the maximum angle ϕ_0 $\dot{\phi} = 0$, which corresponds $E = U$, so

$$E = -mgl \cos\phi_0 \tag{10}$$

Notice that this is a constant value. We calculate now the time required to go from $\phi = 0$ to $\phi = \phi_0$. The period will be 4 times this value. So

$$T = 4\sqrt{\frac{l}{2g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_0}} \tag{11}$$

Using that $\cos\phi = 1 - 2\sin^2\frac{\phi}{2}$

$$T = 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2(\phi_0/2) - \sin^2(\phi/2)}} \tag{12}$$

If we define $\sin\xi = \frac{\sin(\phi/2)}{\sin(\phi_0/2)}$ we get

$$T = 4\sqrt{\frac{l}{g}} K(\sin(\phi_0/2))) \tag{13}$$

where

$$K(k) = \int_0^{\pi/2} \frac{d\xi}{\sqrt{1 - k^2\sin^2(\xi)}} \tag{14}$$

With $K(k)$ an elliptic integral of the first kind. For small oscillations we have $\sin\frac{1}{2}\phi_0 \simeq \frac{1}{2}\phi_0 \ll 1$ and the expansion of K gives

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}{\phi_0}^2 + \cdots\right) \tag{15}$$

which to first order gives

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{16}$$

The reduced mass

In classical mechanics the so called 2 body problem (two particles interacting through a potential energy like the motion of the planets under the attraction of the Sun) can be solved exactly.
This is not true in the theory of general relativity. There is no closed exact solution of the Einstein's equations for the 2 body problem.

But the solution of it within the framework of classical mechanics is the most splendid accomplishment of modernity.

Within the framework of the Lagrangian formalism the solution is extremely elegant and mathematically simple.

The first step consist in breaking down the motion of the system into the motion of the center of mass and the motion of the two bodies around it. The potential energy of the interaction of the two bodies depends only on the distance between them, i.e. on $|\vec{r_1} - \vec{r_2}|$, which we can call it $\vec{r} = \vec{r_1} - \vec{r_2}$. The Lagrangian of the system is:

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(|\vec{r_1} - \vec{r_2}|) \tag{17}$$

And now we can put the origin of our coordinate system at the center of mass of the mechanical system. Then

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 \tag{18}$$

which with the definition

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2 \tag{19}$$

gives

$$\vec{r}_1 = \frac{m_2 \vec{r}}{m_1 + m_2} \qquad \vec{r}_2 = \frac{-m_1 \vec{r}}{m_1 + m_2} \tag{20}$$

Also notice that

$$\begin{aligned}\frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} =& \frac{1}{2}\frac{m_1 m_2^2\dot{\vec{r}}^2}{(m_1+m_2)^2} + \frac{1}{2}\frac{m_1^2 m_2\dot{\vec{r}}^2}{(m_1+m_2)^2} \\ =& \frac{1}{2}m\dot{\vec{r}}^{\,2}\end{aligned} \tag{21}$$

where

$$m = \frac{m_1 m_2}{m_1 + m_2} \tag{22}$$

is called the reduced mass of the system. With this definition the Lagrangian becomes

$$L = \frac{1}{2}m\dot{\vec{r}}^{\,2} - U(r) \tag{23}$$

This is the Lagrangian of one particle moving in an external field $U(r)$ symmetric about the origin (it does not depend on a particular direction).

Once we solve the equations of motion for this $\vec{r}$ using (20) we can find $\vec{r}_1 = \vec{r}_1(t)$ and $\vec{r}_2 = \vec{r}_2(t)$.

Motion in a central field

$U(r)$ in (23) is called a central field. The force acting on the particle is

$$\vec{F} = -\frac{\partial U}{\partial r} = -\frac{dU}{dr}\frac{\vec{r}}{r}$$

As we saw in the previous Lesson notes in the chapter on Angular Momentum, the angular momentum of a system relative to the center of the system is conserved

$$\vec{M} = \vec{r} \times \vec{p}$$

Since $\vec{M}$ is always perpendicular to the radius, this shows that the motion takes place on a plane perpendicular to $\vec{M}$.

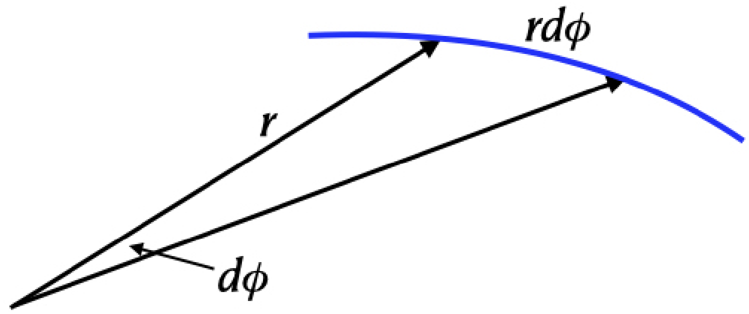

Figure 3: a segment of the path

The path of a particle in a central field lies in one plane.

In polar coordinates the Lagrangian is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r) \tag{24}$$

This Lagrangian does not involve ϕ explicitly, so

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \tag{25}$$

which means

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = p_\phi = M = M_z = constant \tag{26}$$

We can see that $\frac{1}{2}rrd\phi$ is the area of the sector bounded by two adjacent radius vectors and a segment of the path followed by the body (see Fig 3). A differential of the area $df = \frac{1}{2}r^2 d\phi$ divided by a differential of time dt will give us the change of area along the path with time). But precisely with this definition of df

$$M = 2m\frac{df}{dt} = 2m\dot{f} \tag{27}$$

where $\dot{f}$ is called the sectorial velocity, we can see that it is constant, and its constancy implies that the radius vector sweeps equal areas in equal intervals of time (Kepler's second law).

We can construct the solution now using the conservation laws of energy and momentum. We will use the fact that E and M are constants of motion.

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + U(r) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{M^2}{mr^2} + U(r) \tag{28}$$

From here solving for $\dot{r}$ we get

$$\dot{r} \equiv \frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - U(r)\right) - \frac{M^2}{m^2r^2}} \tag{29}$$

and solving for t

$$t = \int dt = \int \frac{dr}{\sqrt{\frac{2}{m}\left(E - U(r)\right) - \frac{M^2}{m^2r^2}}} + constant \tag{30}$$

And solving for dt in $M = mr^2\dot{phi} = mr^2 d\phi/dt$ we can get the differential equation for the trajectory

$$\phi = \int d\phi = \int dt = \int \frac{M/r^2 dr}{\sqrt{\frac{2}{m}\left(E - U(r)\right) - \frac{M^2}{m^2r^2}}} + constant \tag{31}$$

Equations (30) and (31) are the solution to our problem. (30) gives the position (or distance) of our particle. (31) gives $r(\phi)$, namely the path.

Formula (28), is quite important. It shows that the radial component of the motion can be treated as taking place in 1-dimension with an "effective" potential energy

$$U_{eff} = U(r) + \frac{M^2}{2mr^2} \tag{32}$$

$\frac{M^2}{2mr^2}$ is called the centrifugal energy.

The values of r for which

$$U_{eff} = U(r) + \frac{M^2}{2mr^2} = E \tag{33}$$

determine the maximum and minimum effective distance from the center of the field.

Clearly (33) is satisfied only when $\dot{r} = 0$, which is not a stopping point point but a turning one. Before or after this point $r(t)$ increases or decreases.
If the values of r allowed are such that $r \geqslant r_{min}$ the motion is infinite. The particle comes from and returns to infinity.

If If the values of r allowed are such that $r_{min} < r < r_{max}$ movement is finite and the path lies between an annulus bounded by r_{min} and r_{max}, which does not necessarily implies a closed curve.

We can see that between r_{min} and r_{max},

$$\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{M dr/r^2}{\sqrt{2m\left(E - U(r)\right) - M^2/r^2}} \tag{34}$$

For a closed path $\Delta\phi = 2\frac{m}{n}\pi$ with m and n both integers. After n periods the radius vector will complete m revolutions coming back to the original position.
These cases are exceptional. In general $\Delta\phi$ will not be a rational fraction of 2π. There are only two types of $U(r)$ for which the motion takes place on a closed path.
These are

$$U(r) = \frac{1}{r} \qquad \text{or} \qquad U(r) = \frac{1}{r^2} \tag{35}$$

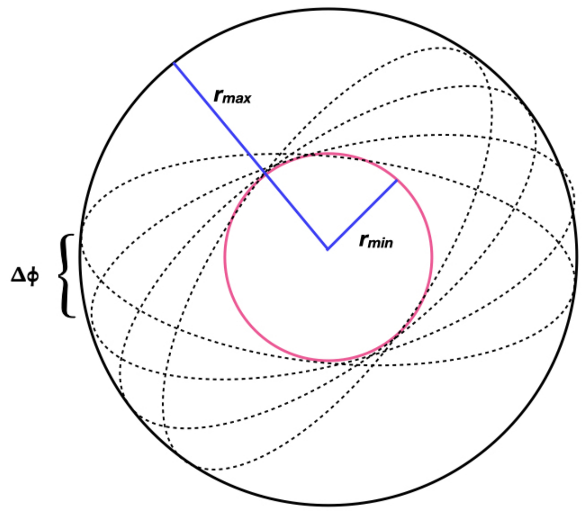

Figure 4: the path under a central field oscillating between a minimum and a maximum radius.

The first is the Kepler's potential. The second one is the space oscillator.

Can the particle reach the center of the field?

When $M \neq 0$ we have the centrifugal energy $1/r^2 \to \infty$ as $r \to 0$. So in general it is impossible.
It could only happen if $U(r) \to -\infty$ as $r \to 0$.

We have from

$$\frac{1}{2}m\dot{r}^2 = E - U(r) - \frac{M^2}{2mr^2} > 0 \tag{36}$$

from where we can infer then that

$$r^2U(r) + \frac{M^2}{2mr^2} < Er^2 \tag{37}$$

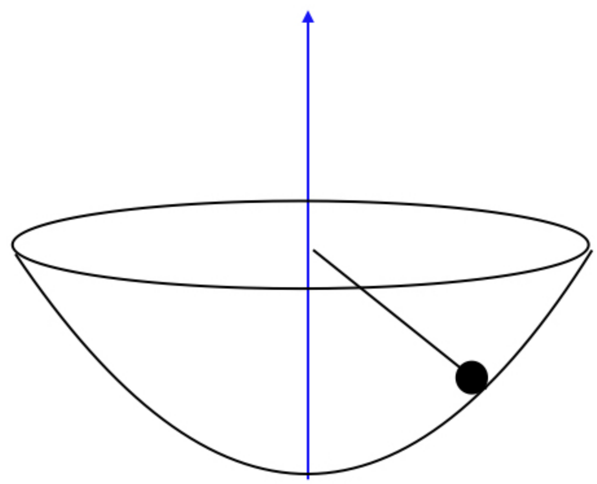

Figure 5: the path for a spherical oscillator.

and consequently $r \to 0$ if,

$$[r^2 U(r)]_{r\to 0} < -\frac{M^2}{2m} \tag{38}$$

which means that $U(r)$ needs to go to $-\infty$ as $-\frac{\alpha}{r^2}$ with $\alpha > \frac{M^2}{2m}$ or proportionally to $-\frac{1}{r^n}$ with $n > 2$.

Example

Integrate the equations of motion for a spherical pendulum (a particle of mass m moving on the surface of a sphere of radius l in a gravitational field).

Solution
The Lagrangian is

$$L = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^{\theta}) + mgl\cos\theta \tag{39}$$

We can see that ϕ is cyclic. So p_ϕ is conserved,

$$M_z = ml^2 \phi \sin^2 \theta = constant \tag{40}$$

The energy is

$$\begin{aligned} E =& \frac{1}{2} ml^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^\theta) - mgl \cos \theta \\ =& \frac{1}{2} ml^2 \dot{\theta}^2 + \frac{1}{2} \frac{{M_z}^2}{ml^2} \sin^2 \theta - mgl \cos \theta \end{aligned} \tag{41}$$

Solving for t in a manner similar to what we did with equation (3)

$$t = \int \frac{d\theta}{\sqrt{2[E - U_{eff}(\theta)/ml^2]}} \tag{42}$$

where $U_{eff} = \frac{1}{2} \left({M_z}^2 / ml^2 \right) \sin^2 \theta - mgl \cos \theta$.

Solving for ϕ

$$\phi = \frac{M_z}{l\sqrt{2m}} \int \frac{d\theta}{\sin^2 \theta \sqrt{E - U_{eff}}} \tag{43}$$

the solution in both cases are elliptic integrals.

Kepler's Potential

One of the most important field types in physics is

$$U(r) \propto \frac{1}{r} \tag{44}$$

where the associated force goes like $1/r^2$. The Kepler's potential energy is of this form as well as the Coulomb's potential in electrostatics. In this latter case the potential could be such that the force between particles is

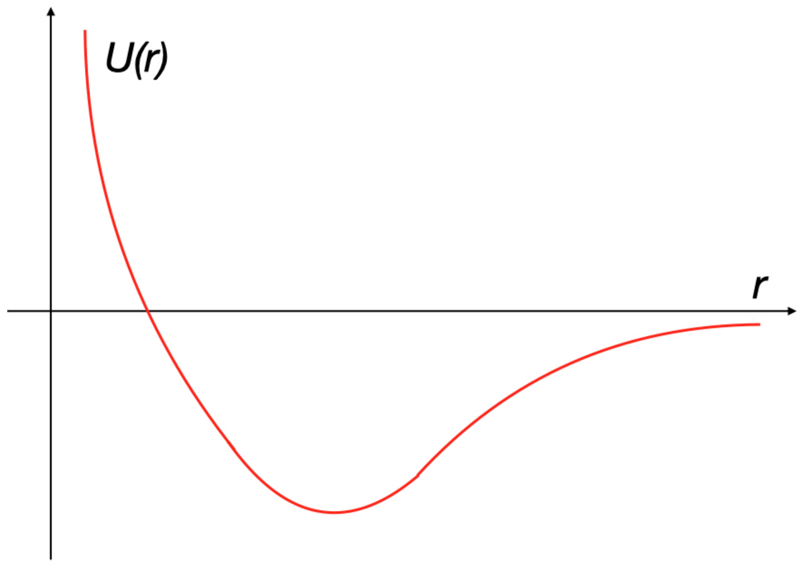

Figure 6: The Kepler effective potential energy

repulsive or attractive depending on the nature of the charge of the particles (charges of equal types repulse each other and opposite charges attract each other). Mass is always positive and the resultant force is attractive only.

The potential we will investigate has the form:

$$U(r) = -\frac{\alpha}{r} \tag{45}$$

The associated effective potential energy is:

$$U_{eff}(r) = -\frac{\alpha}{r} + \frac{M^2}{2mr^2} \tag{46}$$

We can see that when $r \to 0$, $U_{eff} \to \infty$ and when $r \to \infty$, $U_{eff} \to 0$ from negative values of U_{eff}. The minimum of U_{eff} occurs when

$$\frac{dU_{eff}}{dr} = \frac{\alpha}{r^2} - \frac{M^2}{mr^3} = 0 \tag{47}$$

which shows that it occurs when

$$r = \frac{M^2}{m\alpha} \tag{48}$$

and the the effective potential at this value of r is

$$U_{eff} = -\frac{m\alpha^2}{2M^2} \tag{49}$$

We can now use equation (31) where we integrate the trajectory for the angle as function of r. In that equation we substitute the generic $U(r)$ for $U(r) = -\frac{\alpha}{r}$ We then have

$$\int d\phi = \int \frac{M dr/r^2}{\sqrt{2mE - \frac{\alpha m}{r} - \frac{M^2}{r^2}}} \tag{50}$$

from where we have

$$\phi = \frac{M}{\sqrt{2m}} \int \frac{dr/r}{\sqrt{Er^2 - \alpha r - \frac{M^2}{2m}}} \tag{51}$$

We have deliberately reorder the integral to leave the terms inside the square root in the denominator as a polynomial function of r. The integral is now of the form

$$\int \frac{dx/x}{\sqrt{A + Bx + Cx^2}} \tag{52}$$

which has a solution of the form:

$$\int \frac{dx/x}{\sqrt{A + Bx + Cx^2}} = \pm \frac{1}{\sqrt{-A}} \sin^{-1} \left(\frac{Bx + 2A}{x\sqrt{B^2 - 4AC}} \right) \tag{53}$$

We can apply this result in (51) with the following identification:

$$A = -\frac{M^2}{2m}, \qquad B = -\alpha, \qquad C = E$$

Then the integral becomes

$$\phi - \phi_0 = \pm \sin^{-1}\left(\frac{-\alpha r - M^2/m}{r\sqrt{\alpha^2 + 2EM^2/m}}\right) \tag{54}$$

The resulting integral can be simplified further in this manner

$$\phi - \phi_0 = \pm \sin^{-1}\left(\frac{-1 + M^2/\alpha m r}{\sqrt{1 + 2EM^2/m\alpha}}\right) \tag{55}$$

Defining

$$p = \frac{M^2}{m\alpha}, \qquad e = \sqrt{1 + \frac{2EM^2}{m\alpha}} \tag{56}$$

we get

$$\sin\phi = \frac{-1 + p/r}{e} \tag{57}$$

or

$$\frac{p}{r} = 1 + e\sin\phi \tag{58}$$

And finally we can choose the constant ϕ_0 in (55) so that

$$\frac{p}{r} = 1 + e\cos\phi \tag{59}$$

This would give ϕ such that the smallest value of r occurs at $\phi = 0$. This point nearest to the origin is called the perihelion (in the case of an object other than the sun -helios in Greek- it is called periastron). $2p$ is called the *latus rectum* of the orbit and e the eccentricity.

For the obtention of the different formulae involving the major and minor axis and other geometrical parameters of an ellipse see, i.e. [1]. The *latus*

[1] https://www.mathopenref.com/ellipseaxes.html

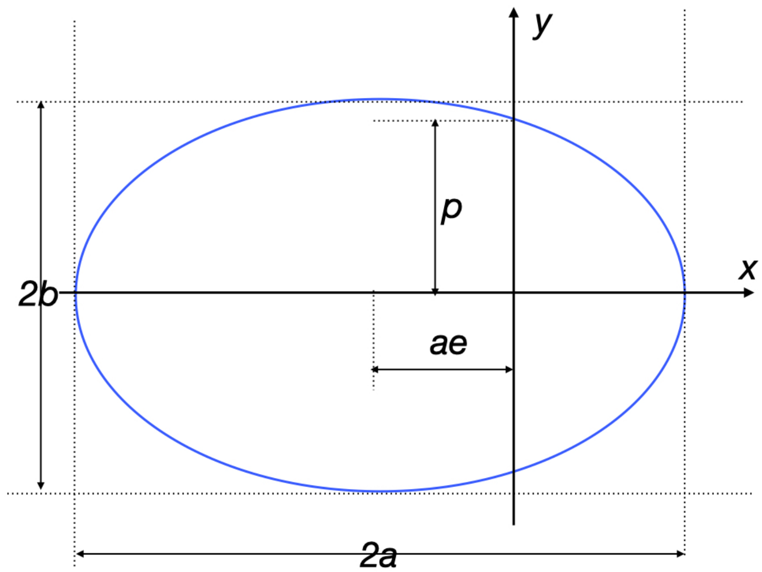

Figure 7: An elliptical orbit and its physical parameters

rectum is given by

$$p = \frac{M^2}{m\alpha} \tag{60}$$

and the eccentricity is

$$e = \sqrt{1 + \frac{2EM^2}{m\alpha^2}} \tag{61}$$

Let's remember that we have made the motion of two bodies bound by a gravity potential equivalent to the motion of one body in a central field. The path of the motion follows a conic. These curves are called conics because they can be described as the result of the intersection of plane with a double cone joined through their vertices. See figure 8. Notice the following:

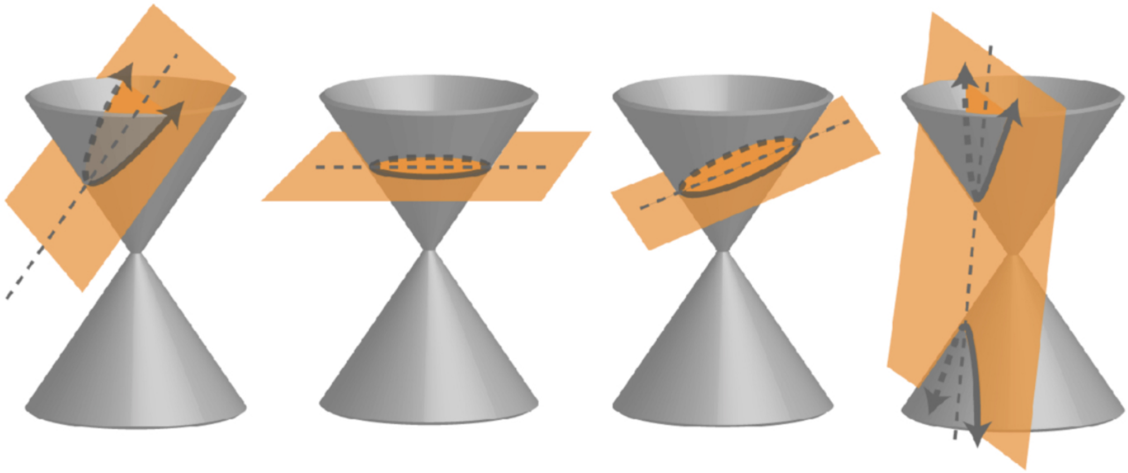

Figure 8: The conics

- If the energy has the value:

$$E = -\frac{m\alpha^2}{2M^2} \tag{62}$$

  $e = 0$ and the orbit is circular.

- If the energy has the value:

$$-1 < -\frac{m\alpha^2}{2M^2} < 0 \tag{63}$$

  the eccentricity is $0 < e < 1$ the orbit is an ellipse.

- If the energy is $E = 0$ the trajectory is a parabola with eccentricity $e = 1$.
  And finally,

- If the energy is $E > 0$ the trajectory is a hyperbola with $e > 1$.

The relationship of the conic parameters to the energy and angular momentum of the involved bodies are:

$$a = \frac{p}{1-e^2} = \frac{\alpha}{2|E|} \qquad\qquad b = \frac{p}{\sqrt{1-e^2}} = M\sqrt{2m|E|} \tag{64}$$

Notice that for bound orbits (like the elliptical and circular ones) the total energy has to be negative or at most 0, i.e. in this last limiting case the kinetic energy is equal to the potential energy, which is always negative. We can also observe that equation (62) is precisely equation (49), i.e. the value of the U_{eff} at the minimum of the potential where motion takes place at fixed value r as it is expected to be a circular orbit. the radius at which the minimum of the potential occurs is

$$r = \frac{M^2}{m\alpha} \tag{65}$$

which we obtained as equation (48) when looking for the minimum of the effective potential. If the kinetic energy exceeds the potential energy the particle would escape the field attraction.

Also we can see that the major axis of the ellipse depends on the energy of the particle and not on its angular momentum.

The least and longest distances from the center of the field are

$$r_{min} = \frac{p}{1+e} = a(1-e) \tag{66}$$

$$r_{max} = \frac{p}{1-e} = a(1+e) \tag{67}$$

These values can be obtained as the roots of

$$U_{eff}(r) = -\frac{\alpha}{r} + \frac{M^2}{2mr^2} = E \tag{68}$$

Using the law of conservation of momentum, i.e. that M in $M = 2mdf/dt$ is a constant, we have

$$\int M dt = \int 2mdf \tag{69}$$

Integrating from 0 to 2π the integral over time is the period of motion

$$MT = \int 2mdf = 2mf = 2\pi mab \tag{70}$$

where the are of the ellipse is $f = 2\pi ab$. From equation (64) we have

$$ab = \frac{p}{1-e^2}\frac{p}{\sqrt{1-e^2}} = \frac{\alpha}{2|E|}\frac{M}{\sqrt{2m|E|}} \tag{71}$$

From where we can obtain using (70)

$$T = \frac{\pi\alpha}{\sqrt{\frac{1}{2}m}\ \ |E|^{3/2}} \tag{72}$$

We observe that the period only depends on the energy of the particle. As we already noticed before for $E \geqslant 0$ motion is infinite, i.e. unbound. In that case the path is a hyperbola with origin at the internal focus: see Figure 9.

The distance to the perihelion is

$$r_{min} = \frac{p}{e+1} = a(e-1) \tag{73}$$

where

$$a = \frac{p}{e^2-1} = \frac{\alpha}{2E} \tag{74}$$

is the axis of the hyperbola.

If instead $E = 0$ we have $e = 1$ and the trajectory is a parabola: see Figure 10. In this case the perihelion is

$$r_{min} = \frac{1}{2}p \tag{75}$$

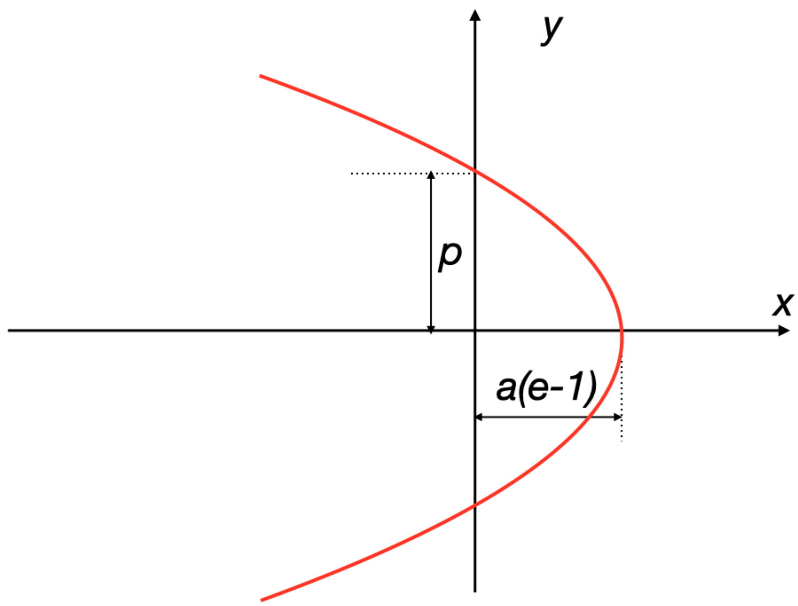

Figure 9: A hyperbolic trajectory

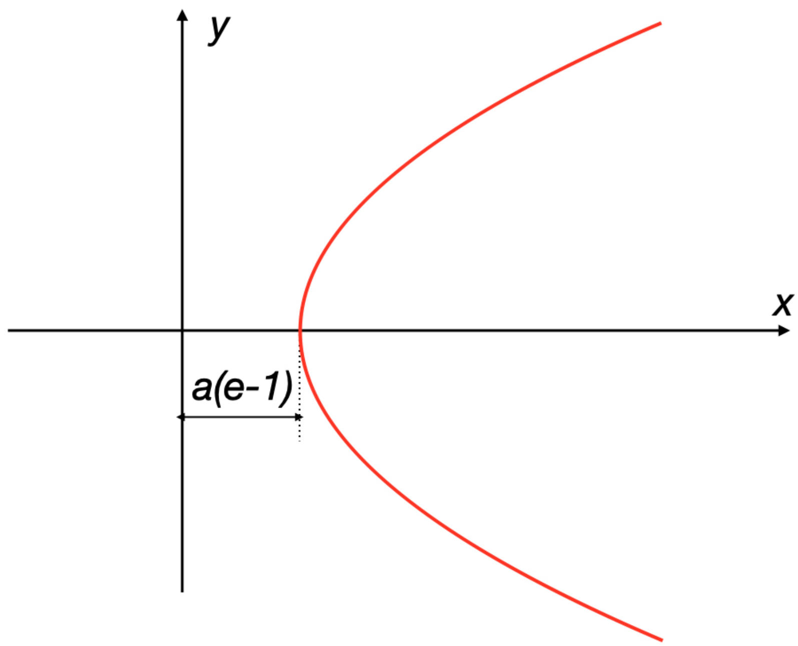

Figure 10: A parabolic trajectory

Finding the coordinates of motion

We need to use equation (30)

$$t = \int dt = \int \frac{dr}{\sqrt{\frac{2}{m}\left(E - U(r)\right) - \frac{M^2}{m^2 r^2}}} + constant$$

where we will use the Kepler potential getting

$$t = \sqrt{\frac{m}{2|E|}} \int \frac{r dr}{\sqrt{-r^2 + \frac{\alpha}{|E|} r - \frac{M^2}{2m|E|}}} = \sqrt{\frac{ma}{\alpha}} \int \frac{r dr}{\sqrt{a^2 e^2 - (r-a)^2}} \tag{76}$$

where e and a are given by equations (61) and (64). Substituting

$$r - a = -ae\cos\xi \tag{77}$$

and then

$$T = \sqrt{\frac{ma^3}{\alpha}} \int (1 - e\cos\xi) d\xi = \sqrt{\frac{ma^3}{\alpha}} (\xi - e\sin\xi) + constant \tag{78}$$

We can take the origin of time so that the constant is 0.

$$r = a(1 - e\cos\xi), \qquad t = \sqrt{\frac{ma^3}{\alpha}} (\xi - e\sin\xi) \tag{79}$$

When $t = 0$ the particle is at the perihelion.
In cartesian coordinates

$$ex = p - r = a(1 - e^2) - a(1 - e\cos\xi) = ae(\cos\xi - e) \tag{80}$$

Using that $y = \sqrt{r^2 - x^2}$ we get

$$x = a(\cos\xi - e) \qquad y = a\sqrt{1 - e^2}\sin\xi \tag{81}$$

and ξ sweeps from 0 to 2π completing the ellipse.
In the case of hyperbolic orbits,

$$r = a(\cos\xi - 1) \qquad t = \sqrt{\frac{ma^3}{\alpha}}(e\sinh\xi - \xi) \tag{82}$$

$$x = a(e - \cosh\xi) \qquad y = a\sqrt{e^2 - 1}\sinh\xi \tag{83}$$

where $-\infty < \xi < \infty$.

A repulsive field

Let's examine the repulsive field

$$U = \frac{\alpha}{r} \qquad \alpha > 0 \tag{84}$$

The effective potential energy will be of the form

$$U_{eff} = \frac{\alpha}{r} + \frac{M^2}{2mr^2} \tag{85}$$

As r goes from 0 to ∞ U_{eff} decreases. The energy of the particle is always positive and motion is then ∞. See Figure 11. The path is a hyperbola

$$\frac{p}{r} - 1 + e\cos\phi \tag{86}$$

where p and e are defined as before.
The perihelion is

$$r_{min} = \frac{p}{e-1} = a(e+1) \tag{87}$$

The time dependence is given

$$r = a(e\cosh\xi + 1) \tag{88}$$

$$t = \sqrt{ma^3\alpha}(e\sinh\xi + \xi) \tag{89}$$

$$x = a(\cosh\xi + e) \tag{90}$$

$$y = a\sqrt{e^2 - 1}\sinh\xi \tag{91}$$

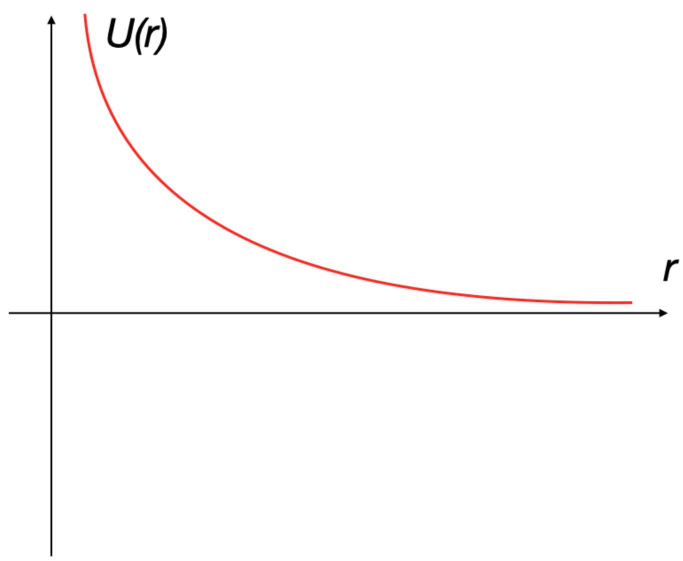

Figure 11: A repulsive central field

One particular integral of motion for these fields

When the field is $U = \alpha/r$ regardless of a positive or negative α, there is one particular integral of motion.
This quantity is

$$\vec{v} \times \vec{M} + \alpha\frac{\vec{r}}{r} \tag{92}$$

and it remains constant throughout motion. Let's calculate the total derivative of it respect to time:

$$\dot{\vec{v}} \times \vec{M} + \alpha\frac{\vec{v}}{r} - \alpha(\vec{v} \cdot \vec{r})\frac{\vec{r}}{r^3} \tag{93}$$

Since $M = m\vec{r} \times \vec{v}$ we have

$$m\vec{r}(\vec{v} \cdot \dot{\vec{v}}) - m\vec{v}(\vec{r} \cdot \dot{\vec{v}}) + \alpha\frac{\vec{v}}{r} - \alpha\frac{r}{r^3}(\vec{v} \cdot \vec{r}) \tag{94}$$

Using that $m\dot{\vec{v}} = \alpha\frac{\vec{r}}{r^3}$ we obtain

$$\frac{d}{dt}(\vec{v} \times \vec{M} + \alpha\frac{\vec{r}}{r}) = 0 \tag{95}$$

The direction is along the major axis from the focus to the perihelion and its magnitude is αe.