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Lesson 2: Conservation Laws

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The generalized coordinates q_i and velocities $\dot{q}_i$ clearly change during the motion experienced by a mechanical system. There are nonetheless certain functions of these quantities that could, under certain circumstances, remain unchanged depending only on the initial conditions of the problem. Such functions are called the *integrals of motion*. **There are $2n - 1$ integrals of motion for a system with n degrees of freedom.**

Let's recap the above statement:
A more proper way of framing it is the following. Think that the utilization of the generalized coordinates and generalized velocities, let us study the motion of a mechanical system in a 2n dimensional space defined precisely by them: this is called a phase space. A constant of motion is a function

$$C(q_i(t_1), \dot{q}_i(t_1), t_1) = C(q_i(t_2), \dot{q}_i(t_2), t_2)$$

On the other hand an integral of motion is a function $I(q_i, \dot{q}_i)$ which remains constant along any trajectory:

$$I(q_i(t_1), \dot{q}_i(t_1)) = I(q_i(t_2), \dot{q}_i(t_2))$$

Notice that $I(q_i, \dot{q}_i)$ it is not a function of time! But integrals of motion are constants of motion. The reciprocal is not true: there are constants of motion that are not integrals of motion, i.e.:

$$C(q, \dot{q}, t) = q - \dot{q}t$$

is a constant of motion but not an integral of motion (it is dependent on t!).

These examples show that using the fact that the integrals of motion do not depend on time, the $2n$ constants of motion can be solved for as function of the generalized coordinates and an arbitrary time. This arbitrariness in the choice of time effectively constrains the $2n$ constants in one (the arbitrary constant time). The general solution of the equations of motion has $2n$ arbitrary constants. Our variables are q_i and $\dot{q}_i$ but not the time explicitly. This makes the choice of the origin of time completely arbitrary. Furthermore we can add an additive constant t_0 to the time and express the $2n$ functions q_i and $\dot{q}_i$ as functions of the arbitrary constants, i.e. we can have $q_i = q_i(t + t_0, C_1, C_2, ..., C_{2n-1})$ and $\dot{q}_i = \dot{q}_i(t + t_0, C_1, C_2, ..., C_{2n-1})$ expressing our generalized coordinates and velocities functions as a function of the $2n - 1$ arbitrary constants.

There are certain integrals of motion that have a distinct relevance though. These are related to symmetries present in the physical system under consideration.

Energy

Homogeneity of time
Homogeneity of time only means that the results of a physical experiment (or effect) do not depend on any time in particular. If I drop a stone from a cliff today, the result will not change by dropping the same stone from the same cliff in a few days from today (assuming all other conditions equal).

This is expressed by the fact that the Lagrangian of a closed physical system does not depend explicitly of time. So let's calculate the total derivative of the Lagrangian respect to time and make it equal to zero. First the total derivative of L is:

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \tag{1}$$

Using E-L equations we have that $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$ and then replacing in (1)

$$\begin{aligned} \frac{dL}{dt} &= \sum_i \dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \\ &= \sum_i \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) \end{aligned}$$

And we can enclosed both sides of the equation in one total derivative respect to time

$$\frac{d}{dt}\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) = 0 \tag{2}$$

We will call the quantity within the parenthesis, which remains constant in time during the motion of a closed system, the Energy of the mechanical system.

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \tag{3}$$

We can notice that E is linear in L and consequently it is additive like the Lagrangian for a system of particles which are not interacting. We notice that this quantity remains constant even in the presence of a constant field if it does not change with time. Mechanical systems where the energy is conserved are called conservative systems.

One more thing to notice. The shape of (3) is such that if we write the Lagrangian as $L = T(q, \dot{q}) - U(q)$ where the function T is a quadratic function of the velocities, we obtain

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T \tag{4}$$

where the last equation is a result of Euler's theorem for homogeneous functions (T is a homogeneous function of degree 2 of $\dot{q}$).

Homogeneous functions

Homogeneous real-valued function: of two variables x and y is a real-valued function that satisfies the condition $f(rx, ry) = r^k f(x, y)$ for some constant k and all real numbers r. The constant k is called the degree of homogeneity.

Euler's homogeneous function theorem

Suppose that the function $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ is continuously differentiable. Then f is positively homogeneous of degree k if and only if $\mathbf{x} \cdot \nabla f(\mathbf{x}) = kf(\mathbf{x})$. [a]

Example

$f(x, y) = 3x + y$

$$\begin{aligned} f(\beta x, \beta y) &= 3\beta x + \beta y = \beta(3x + y) \\ &= \beta^1(3x + y) = \beta^1 f(x, y) \end{aligned} \tag{5}$$

[a] for a proof see https://en.wikipedia.org/wiki/Homogeneous_function#Euler's_theorem

Substituting in (3) we get

$$E = T(q, \dot{q}) + U(q); \tag{6}$$

In cartesian coordinates for a system on n masses,

$$E = \sum_i \frac{1}{2} m_i {v_i}^2 + U(\vec{r_1}, \vec{r_2}, ..., \vec{r_n}) \tag{7}$$

Where i goes from 1 to n, and the potential depends only on the position of the masses. The total energy of a mechanical system is the sum of the kinetic energy T and the potential energy U.

Momentum

Let's consider now the homogeneity. What does this mean? If we think about a closed mechanical system we always model it as a system in some sort of ideal vacuum space. We neglect air or any medium. This imaginary model "vacuum" it's in some sense of constant density. We don't think that this hypothetical medium changes as we move through it in any direction. This is what we mean by a homogeneous space. If we perform a displacement in any direction, the mechanical state of the system, where each particle making our system is moved similarly in the same direction, should not change. The Lagrangian is changing only due to these displacement of the system, so

$$\delta L = \sum_i \frac{\partial L}{\partial \vec{r}_i} \cdot \delta \vec{r}_i = 0 \tag{8}$$

due to the fact that $\delta \vec{r}_i$ is arbitrary we get

$$\sum_i \frac{\partial L}{\partial \vec{r}_i} = 0 \tag{9}$$

We can use E-L equations (Lesson 1)

$$\frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \vec{v}_i}\right) - \sum_i \frac{\partial L}{\partial \vec{r}_i} = 0 \tag{10}$$

where $\vec{v}_i = \dot{\vec{r}}_i$ from where we clearly obtain that

$$\frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \vec{v}_i}\right) = \sum_i \frac{\partial L}{\partial \vec{r}_i} = 0 \tag{11}$$

This means that in a closed mechanical system, the vector

$$\vec{P} \equiv \sum_i \frac{\partial L}{\partial \vec{v}_i} = constant \tag{12}$$

$\vec{P}$ remains constant throughout motion and it is called the momentum of the system. The Lagrangian for a system of particles is

$$L = \sum_{i=1}^{n} \frac{1}{2} m_i {v_i}^2 - U(\vec{r_1}, \vec{r_2}, ..., \vec{r_n}) \tag{13}$$

So for a system of particles

$$\vec{P} = \sum_i \frac{\partial L}{\partial \vec{v_i}} = \sum_i m_i \vec{v_i} = \sum_i \vec{p_i} \tag{14}$$

Expressing the additivity of the momenta of a system particles. This is true wether their exists interaction between them or not. All three components of momentum will be conserved only if there is no external field. In the presence of one some components may be conserved if the potential energy does not depend on the corresponding coordinates.
Example:
If $\frac{\partial U}{\partial x_1} = 0$ then $P_x = \sum_i m_i v_{ix}$ remains constant throughout the motion of the system. In other words displacement in the x_i direction does not change the momentum of the system. Or vice versa and using cartesian coordinates if the field is uniform in the x direction then the momentum of the system remains unchanged in the y and z direction. Also we note that

$$\frac{\partial L}{\partial \vec{r_i}} = -\frac{\partial U}{\partial \vec{r_i}} \tag{15}$$

is the force $\vec{F_i}$ acting on the i particle and this implies from eq (8) that the sum of all forces acting on all the particles in the system is

$$\sum_i \vec{F_i} = 0 \tag{16}$$

We can see that if the system is made out of just two particles

$$\vec{F_1} + \vec{F_2} = 0 \tag{17}$$

from where $\vec{F_1} = -\vec{F_2}$ which means that the force exerted by particle 1 on particle 2 is equal in magnitude and opposite in direction to the force exerted by 2 on 1 (Newton's third law, action and reaction).

Generalized Coordinates

When using generalized coordinates q_i, we have that the derivatives of the Lagrangian, with respect to the generalized velocities

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{18}$$

are called the generalized momenta. And the Lagrangian derivative respect to the generalized coordinates

$$F_i = \frac{\partial L}{\partial q_i} \tag{19}$$

are called generalized momenta. Notice then that with this notation the E-L equations are

$$\dot{p}_i = F_i \tag{20}$$

In Cartesian coordinates the generalized momenta are the components of the vectors $\vec{p_i}$ but in generalized coordinates the generalized momenta will not necessarily be the products of mass and velocities.

Center of mass

Obviously the momentum of a system depends on which frame it is calculated. In non-relativistic cases if a frame O moves with velocity $\vec{V}$ respect to another system O' the velocities of each particle (from the system of particles under consideration) in each frame being $\vec{v_i}$ and $\vec{v_i}'$ respectively, are related by

$$\vec{v}_i = \vec{v}'_i + \vec{V} \tag{21}$$

The momenta are then

$$\vec{P} = \sum_i m_i \vec{v}_i = \sum_i m_i \vec{v}'_i + \sum_i m_i \vec{V} = \sum_i m_i \vec{v}'_i + \vec{V} \sum_i m_i \tag{22}$$

or

$$\vec{P} = \vec{P}' + \vec{V} \sum_i m_i \tag{23}$$

As we discussed before let's be O' the references frame where our system has momentum $\vec{P}' = 0$. From it

$$\vec{P} = \vec{V} \sum_i m_i \tag{24}$$

it follows

$$\vec{V} = \frac{\vec{P}}{\sum_i m_i} = \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i} \tag{25}$$

When the total momentum of a system in a given frame is 0 it is said to be at rest in such frame in analogy with a particle at rest in a frame co-moving with the particle.

Additionally we see that the momentum of the system of particles in the frame where it is at rest can be written as (23) where $\sum_i m_i$ is the sum of the total particles and can be treated as the mass of the system. This shows the additivity of mass.

Additionally (23) can be seen as

$$\vec{V} = \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i} = \frac{\sum_i m_i \frac{d\vec{r}_i}{dt}}{\sum_i m_i} \tag{26}$$

We can now define a vector $\vec{R}$ such that $\frac{d\vec{R}}{dt} = \vec{V}$ which will then be

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \tag{27}$$

This position vector is called the center of mass. Notice that the system then moves with a velocity $\vec{V}$ that it is the time derivative of a very particular point in the system, its center of mass.

It is clear then, that if the linear momentum of the system is conserved the center of mass moves in a straight line with constant velocity. This is the generalization of the principle of inertia. All the particles in the system could be moving in different directions but the system as a whole moves in such a way that the center of mass remains moving with constant velocity like a single particle in the absence of an external potential (force).

Internal energy

The total energy of a system of particles in a given frame O

$$E = \frac{1}{2}\sum_i m_i {v_i}^2 + U \tag{28}$$

On a different system O' moving with velocity $\vec{V}$ respect to O using Galilean transformations of the velocities

$$\begin{aligned} E &= \frac{1}{2}\sum_i m_i (\vec{v}_i' + \vec{V})^2 + U \\ &= \frac{1}{2}\sum_i m_i V^2 + \frac{1}{2}\sum_i m_i \vec{v'}_i^2 + \sum_i m_i \vec{v}_i' \cdot \vec{V} + U \\ &= E' + \vec{V} \cdot \vec{P'} + \frac{1}{2}\mu V^2 \end{aligned} \tag{29}$$

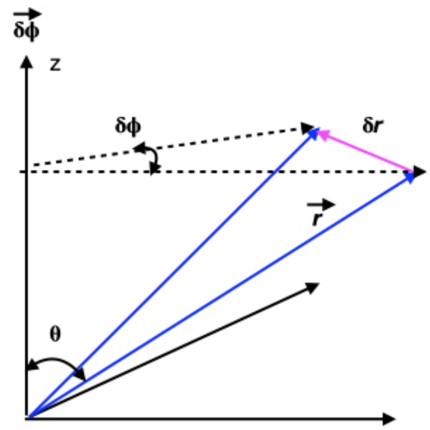

Figure 1: An infinitesimal rotation around the z axis (the direction of $\delta\vec{\phi}$).

where $E' = \frac{1}{2}\sum_i m_i \vec{v'}_i^2 + U$, $\mu = \sum_i m_i$, and $\vec{P'} = \sum_i m_i \vec{v}'_i$. This is the law of transformation of energy between inertial frames in relative motion. E' is the energy of the system in the O' system of reference. If O' is a system of reference comoving with the centre of mass then $\vec{P'} = 0$, and E' is called the internal energy of the system E_i.

Angular momentum

The next conservation principle is related to the isotropy of space. This principle states that the mechanical properties of a physical system do not change it it undergoes a rotation in space. To infer the implications of this principle we will calculate how the Lagrangian transforms under an infinitesimal rotation of the system. We will use a vector $\delta\vec{\phi}$ whose direction corresponds to the axis of rotation. Its magnitude is the angle of rotation $\delta\phi$. We will use spherical coordinates to make the calculation. To visualize it in cartesian coordinates, we can think that the direction of rotation corresponds to the z axis.

We notice that performing a rotation only (and not i.e., a rotation combined with a translation) $\delta\vec{r}$ is necessarily perpendicular to $\delta\vec{\phi}$ and $\delta\vec{r}$. So we have

$$\delta\vec{r} = \delta\vec{\phi} \times \vec{r} \tag{30}$$

We can see from the figure Fig 1 that this is consistent with

$$|\delta\vec{r}| = r \sin\theta \delta\phi \tag{31}$$

We also notice that under a rotation the direction of $\vec{r}$ changes and consequently, as we learn in introductory physics, there is a change in the direction of the velocity.

$$\delta\vec{v} = \delta\vec{\phi} \times \vec{v} \tag{32}$$

We can now calculate the change in the Lagrangian due to these infinitesimal changes. We will impose, following our principle that the Lagrangian does not change under this rotation.

$$\delta L = \sum_i \left(\frac{\partial L}{\partial \vec{r}_i} \cdot \delta\vec{r}_i + \frac{\partial L}{\partial \vec{v}_i} \cdot \delta\vec{v}_i \right) = 0 \tag{33}$$

With $\vec{p}_i = \frac{\partial L}{\partial \vec{v}_i}$ and $\frac{\partial L}{\partial \vec{r}_i} = \dot{\vec{p}}_i$ and using (29), we can write this variation,

$$\delta L = \sum_i \left(\dot{\vec{p}}_i \cdot \delta\vec{\phi} \times \vec{r}_i + \vec{p}_i \cdot \delta\phi \times \vec{v}_i \right) = 0 \tag{34}$$

Remember the properties of the mixed product:

$$\begin{aligned} \vec{a} \cdot (\vec{b} \times \vec{c}) &= \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = (\vec{b} \times \vec{c}) \cdot \vec{a} \\ &= (\vec{c} \times \vec{a}) \cdot \vec{b} = (\vec{a} \times \vec{b}) \cdot \vec{c} \end{aligned}$$

This means that we can write

$$\begin{aligned}\delta L &= \sum_i \left(\vec{\delta\phi} \cdot \vec{r_i} \times \dot{\vec{p_i}} + \vec{\delta\phi} \cdot \vec{v_i} \times \vec{p_i}\right) \\ &= \vec{\delta\phi} \cdot \sum_i \left(\vec{r_i} \times \dot{\vec{p_i}} + \vec{v_i} \times \vec{p_i}\right) \\ &= \vec{\delta\phi} \cdot \frac{d}{dt} \sum_i \vec{r_i} \times \vec{p_i} = 0\end{aligned} \tag{35}$$

Of course $\vec{\delta\phi}$ is arbitrary and not 0 which means that $\frac{d}{dt}\sum_i \vec{r_i} \times \vec{p_i} = 0$. This means that for a closed system the quantity $\sum_i \vec{r_i} \times \vec{p_i}$ remains constant after a rotation. We call it the angular momentum of the system:

$$\vec{M} \equiv \sum_i \vec{r_i} \times \vec{p_i} \tag{36}$$

Like the linear momentum, the angular momentum is also additive regardless of the motion of the particles in the system.

There are no other integrals of motion for an isolated mechanical system, so we have seven integrals of motion: the energy, the three space components of the linear momentum and the three components of the angular momentum. Notice that the value of the angular momentum will depend strongly on the system it is calculated from.

We can now, as we did with the linear momentum, calculate the value of the angular momentum from two inertial systems in relative motion with respect to each other. Let's say we do it from a system where one vector position of the system is $\vec{r}$ and another one from where it is $\vec{r} = \vec{r'} + \vec{a}$

$$\begin{aligned}\vec{M} &= \sum_i \vec{r}_i \times \vec{p}_i \\ &= \sum_i \vec{r'}_i \times \vec{p}_i + \vec{a} \times \sum_i \vec{p}_i \\ &= \vec{M}' + \vec{a} \times \vec{P}\end{aligned} \tag{37}$$

Notice from this that the value of the angular momentum will depend strongly on the system it is calculated from. Except when the system is at rest as a whole (i.e. $\vec{P} = 0$ in which case we have $\vec{M} = \vec{M}'$. Of course, the fact that it's value is relative to the system from where it's observed is still consistent with its conservation.

We will assume now that the two systems O and O' differ also by a relative velocity $\vec{V}$. We can assume for simplification that O and O' coincide at some instant of time. Then the position vectors of the system's particles will be the same and their velocities are related $\vec{v}_i = \vec{v'}_i + \vec{V}$

$$\begin{aligned}\vec{M} &= \sum_i m_i \vec{r}_i \times \vec{v}_i = \sum_i m_i \vec{r}_i \times \vec{v'}_i + \sum_i m_i \vec{r}_i \times \vec{V} \\ &= \vec{M}' + \mu \vec{R} \times \vec{V}\end{aligned} \tag{38}$$

where we have use the definition of center of mass $\vec{R} = \sum_i m_i \vec{r}_i / \mu$ where μ is the total mass of the system.

If O' is the frame in which the system is considered at rest as a whole then $\vec{V}$ is the velocity of its center of mass and the total momentum relative to O is $\vec{P} = \mu \vec{V}$ and we have

$$M = \vec{M}' + \vec{R} \times \vec{P} \tag{39}$$

where we see that the angular momentum is made up of an intrinsic angular momentum in a frame with respect to which is at rest plus the angular momentum due to the motion of the system as a whole $\vec{R} \times \vec{P}$.

We may have situations in which the angular momentum is not conserved completely (i.e. all three components of it). But depending on the symmetry of the external field a particular component could, i.e. the one of the direction of the symmetry of the field, because in this case a rotation around that axis will not change the state of the system.

Examples

In the case of a central field, where its effect depends on the distance of the system to the center of it, any rotation around an axis perpendicular to the field direction will not change the angular momentum of the system provided that it is defined with respect to the center of the field.

In the case of a homogeneous field in the z direction, a rotation around it will not change it either.

This is the proof: The z component of the angular momentum is

$$M_z = \sum_i m_i p_{i_z} = \sum_i m_i((x_i \dot{y}_i - y_i \dot{x}_i) \tag{40}$$

But using cylindrical coordinates this is

$$M_z = \sum_i m_i {r_i}^2 \dot{\phi}_i \tag{41}$$

But the Lagrangian in cylindrical coordinates is

$$L = \frac{1}{2} \sum_i m_i ({r_i}^2 + r_i \dot{\phi}_i^2 + \dot{z}_i^2) - U \tag{42}$$

Clearly this is

$$M_z = \sum_i \frac{\partial L}{\partial \dot{\phi}_i} \tag{43}$$

as (4) shows.

Mechanical similarity

We saw in Lesson 1 that multiplication of a Lagrangian by a constant does not affect the equations of motion. We could use this fact to try to infer properties of the motion of a particular system without fully integrating the equations of motion.

One such a case is when the potential energy is a homogeneous function of the coordinates.

In our case we will have

$$U(\alpha\vec{r}_1, \alpha\vec{r}_2, ..., \alpha\vec{r}_n) = \alpha^k U(\vec{r}_1, \vec{r}_2, ..., \vec{r}_n) \tag{44}$$

where α is a constant and k is the degree of homogeneity of U.

We will try the following transformation: multiply the coordinates by a factor α and the time by a factor β: $\vec{r}_i \to \alpha\vec{r}_i$ and $t \to \beta t$. All the velocities are changed then by a factor α/β and the kinetic energy by α^2/β^2. If α and β are such $\alpha^2/\beta^2 = \alpha^k$, which means $\beta = \alpha^{1-\frac{1}{2}k}$, the net result is to multiply the Lagrangian by a factor α^k and the equations of motion do not change.

What is the meaning of such a transformation? Changing all the coordinates by the same factor is equivalent to replacing the paths to be followed by the system for similar ones, just of a different size.

If the potential energy is a homogeneous function of degree k in the Cartesian coordinates, the equations of motion permit a series of geometrically similar paths, and the times of the motion between corresponding points are given by

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-\frac{1}{2}k} \tag{45}$$

where $\left(\frac{l'}{l}\right)$ is the ratio between the linear dimensions of equivalent paths under the transformation. Other physical quantities are also equivalente under the transformation, i.e. the velocities, energies and angular momentum are,

$$\frac{v'}{v} = \left(\frac{l'}{l}\right)^{\frac{1}{2}k} \qquad \frac{E'}{E} = \left(\frac{l'}{l}\right)^{k} \qquad \frac{M'}{M} = \left(\frac{l'}{l}\right)^{1+\frac{1}{2}k} \tag{46}$$

Examples

1) For small oscillations, as we know from previous courses of introductory mechanics, the potential is of the form $U \propto r^2$. We can see from (46) that the period of oscillation is independent of the amplitude.

2) In a uniform field of force the potential energy is a linear function of the coordinates (the force is constant). In these cases $k = 1$ and (46) leads to $\frac{t'}{t} = \left(\frac{l'}{l}\right)^{\frac{1}{2}}$. This means that the time of motion for two different paths goes like the square root of the ratio of distances travel. In the case of free fall this applies to the time of fall from a given altitude.

3) In the case of a Newton's (or Coulomb's) potential $k = -1$ and we obtain from (46) $\frac{t'}{t} = \left(\frac{l'}{l}\right)^{\frac{3}{2}}$ which is precisely Kepler's third law (the square of the ratio of the periods of revolution between two orbits is equal to the cube of the ratio of their sizes.

The Virial Theorem of Classical Mechanics

If the potential energy is a homogeneous function of the coordinates and the motion takes place in a finite region of space, there is a simple relationship between the average values of the kinetic and potential energies, which plays a prominent role in Astronomy, called the *V*irial Theorem.

If we apply the Euler's theorem to the kinetic energy of a physical system, due to the fact that $T = T(\vec{v}^2)$, we get $\sum_a \mathbf{v}_a \cdot \partial T/\partial \mathbf{v}_a = 2T$ we can write:

$$2T = \sum_a \mathbf{p}_a \cdot \mathbf{v}_a = \frac{d}{dt}\left(\sum_a \mathbf{p}_a \cdot \mathbf{r}_a\right) - \sum_a \mathbf{r}_a \cdot \dot{\mathbf{p}}_a \tag{47}$$

We can now average this equation with respect to time. The average value of a function of time can be defined:

$$\bar{f} = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau f(t)dt$$

But if $f(t)$ is the time derivative $dF(t)/dt$ of a bounded function $F(t)$,

$$\bar{f} = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \frac{dF}{dt}dt = \lim_{\tau\to\infty} \frac{F(\tau) - F(0)}{\tau} = 0$$

If we now look at equation (47) $\sum_a \mathbf{p}_a \cdot \mathbf{r}_a$ is bounded (positions remain within a given region and momenta are not infinite) so when we take the average of $2T$ the first term on the right hand side is 0. For the second term we can replace $\dot{\mathbf{p}}_a$ by $-\partial U/\partial \mathbf{r}_a$ following Newton's second law and get:

$$2\bar{T} = \overline{\sum_a \mathbf{r}_a \cdot \partial U/\partial \mathbf{r}_a} \tag{48}$$

Using now Euler's theorem:

$$2\bar{T} = k\bar{U} \tag{49}$$

Since $\bar{T} + \bar{U} = \bar{E} = E$ (energy is conserved!), (49),

$$\bar{U} = 2\frac{E}{k+2}, \qquad \bar{T} = \frac{k}{k+2}E \tag{50}$$

expressing $\bar{U}$ and $\bar{T}$ in terms of the total energy of the system.

For small oscillations $k = 2$ and we get $\bar{U} = E/2$ and $\bar{T} = E/2$

$$\bar{T} = \bar{U} \tag{51}$$

Due to the fact that in Newton's gravitational potential is $k = -1$ (49), leads as well to:

$$\bar{U} = 2E \ , \ \bar{T} = -E \tag{52}$$

Showing that in this type of interaction motion of a system is bounded (limited to occupy a finite region) only if the energy is negative.

In Astronomy $\overline{\sum_a \mathbf{r}_a \cdot \partial U / \partial \mathbf{r}_a}$ is called the *virial* of the system.

A unified approach to conservation laws: Noether's theorem

The formalism developed studying the constants of motion in Lagrangian mechanics, which we discussed in the previous sections, was developed in 1788 and a variation of it using the Hamiltonian formalism, which we will discuss later on, was developed in 1883.
In 1918, Emily Noether, provided a uniform framework which was more general and mathematically very elegant, proving what now is called Noether's first theorem:

Every differentiable symmetry of the action of a physical system with conservative forces has a corresponding conserved quantity.

Let's try to understand the meaning of this theorem. We need a few definitions. First we define the meaning of conserved quantity:

Conserved quantity

From the principle of least action we found that the action

$$S(q_i(t)) = \int_L (q_i, \dot{q}_i, t)dt \tag{53}$$

is an extreme(minimum) when $q_i(t)$ satisfies the Euler-Lagrange equation:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \tag{54}$$

As we already studied this condition guarantees that δS vanishes for all variations $q_i(t) \to q_i(t) + \delta q_i(t)$ which are zero at the beginning and end of the trajectory.

Let's assume then that $q_i(t)$ is a solution of (54). We will called any function $f = f(q_i(t), \dot{q}_i(t)$, that fulfills the condition that

$$\frac{df}{dt}_{q_i(t)} = 0 \tag{55}$$

along the actual path of motion, a conserved quantity.

Symmetry of the action

A symmetry of the action is any transformation of the path $q_i(t) \to \lambda_i(q_j(t), t)$ that leaves the action invariant.

$$S[q_i(t)] = S[\lambda_i(q_j(t), t)] \tag{56}$$

and if $\lambda_i(q_j)$ represents a continuous transformation of q_i, we can expand the transformation to keep at first order

$$q_i \to q'_i = q_i + \epsilon_i(q_j) \tag{57}$$

and where δq_i obviously is

$$\delta q_i = q'_i - q_i = \epsilon_i(q_j) \tag{58}$$

and with this an infinitesimal change in the action is

$$\delta_\epsilon S = S[q_i + \epsilon_i(q_j)] - S[q_i] = 0 \tag{59}$$

Notice that in principle, this is true regardless of the fact that $q_i(t)$ satisfies the E-L equations or not. Additionally if the action is invariant under an infinitesimal symmetry, we can perfrom several of them, provided the transformations are smooth mathematically, and obtain a finite symmetry transformation. We can now look at Noether's theorem:

Suppose the action depends on n functions $q_i(t)$ has a symmetry such that

$$\delta q_i = {q'}_i - q_i = \epsilon_i(q_j) \tag{60}$$

Then the quantity

$$I = \frac{\partial L(q_i(\lambda))}{\partial \dot{q}_i}\epsilon_i(q_j) \tag{61}$$

is conserved.

Proof

The existence of a symmetry implies

$$\begin{aligned}\delta_\epsilon S[q(t)] \equiv& 0\\ &\equiv \sum_{i=1}^{N} \int \left(\frac{\partial L(q_i(t),\dot{q}_i))}{\partial q_i}\epsilon_i(q_j) + \frac{\partial L(q_i(t),\dot{q}_i))}{\partial \dot{q}_i}\frac{d\epsilon_i(q_j)}{dt}\right) dt\end{aligned} \tag{62}$$

We can integrate the second term by parts

$$\begin{aligned}0 =& \int \left(\frac{\partial L(q_i(t),\dot{q}_i))}{\partial q_i}\epsilon_i(q_j)\right) + \frac{d}{dt}\left(\frac{\partial L(q_i(t),\dot{q}_i)}{\partial \dot{q}_i}\epsilon_i(q_j)\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\epsilon_i(q_j)\right) dt\\ =&\frac{\partial L}{\partial \dot{q}_i}\epsilon_i(q_j)\bigg|_{t_1}^{t_2} + \int \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)\epsilon_i(q_j)dt\end{aligned} \tag{63}$$

This is an expression valid for every possible path. Let's assume now that the path obeys the principle of least action, i.e. our system equations of motion are Euler Lagrange's equations of motion (54):

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \tag{64}$$

Then the integral in equation (63) vanishes and we have

$$\begin{aligned} \delta S(q_i) =& \frac{\partial L}{\partial \dot{q}_i}\epsilon_i(q_j)\Big|_{t_1}^{t_2} \\ =& I(t_2) - I(t_1) = 0 \end{aligned} \tag{65}$$

where

$$I = \frac{\partial L(q_i, \dot{q}_i)}{\partial \dot{q}_i}\epsilon_i(q_j) \tag{66}$$

From (65) we have then that

$$\frac{dI}{dt} = 0 \tag{67}$$

So I is a constant of the motion.