Classical Mechanics 2022
Lesson 1: Generalized Coordinates and the equations of motion

Mario C Díaz

Classical Covariance

This course is fundamentally a study of the mechanics of point like particles (material points). We will not be studying the mechanics of the continuum or fluids. The foundations developed in this course of mechanics are essential to pursue those studies for those of you who are interested, though.

To understand the approach we will take in the study of mechanics it is interesting to reflect on the meaning of invariance of the law of physics. Diffeomorphism invariance is one the most fundamental concepts introduced by the modern physics of the XX century. Its first incarnation was actually a product of Galilean relativity. Diffeomorphism[1] invariance, simply means "invariance of the laws of physics between coordinate systems." It can also be called coordinate covariance, or just covariance. This idea underlies the foundations of physics as we currently understand it. The invariance in the description of motion from inertial systems in relative motion which each other can be called classical covariance. In the special theory of relativity we have special covariance, while in the general theory of relativity we talk of general covariance, namely that the law of physics are the same even for non-inertial systems of reference. Simply put this means: the use of any type of coordinate system does not affect the physics experienced by different observers. A specific physics calculation applied to some problem should give exactly the same answers regardless the use of any particular coordinate system (Cartesian, spherical, cylindrical, etc).

On the other hand it is true that depending on the coordinates used we could get what at first glance look very dissimilar equations representing the same physical situation. For example Newton's equations,

$$\vec{F} = m\vec{a} \tag{1}$$

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[1]In mathematics, a diffeomorphism is an invertible function that maps one differentiable manifold (a "space" where there are no singularities) to another such that both the function and its inverse are differentiable.

in 2-D cartesian coordinates becomes:

$$F_x = ma_x \tag{2}$$
$$F_y = ma_y \tag{3}$$

or more explicitly

$$F_x = m\frac{d^2x}{dt^2} \tag{4}$$
$$F_y = m\frac{d^2y}{dt^2} \tag{5}$$

or

$$F_x = m\ddot{x} \tag{6}$$
$$F_y = m\ddot{y} \tag{7}$$

But if we use polar coordinates r, θ we can not write

$$F_r = m\ddot{r} \tag{8}$$
$$F_\theta = m\ddot{\theta} \tag{9}$$

(8) and (9) are NOT the equations of motion of our system. Using the relationship existing between the coordinates:

$$x = r\cos\theta \tag{10}$$
$$y = r\sin\theta \tag{11}$$

We have a transformation law between both systems and we can obtain

$$F_r = m\ddot{r} - mr\dot{\theta}^2 \tag{12}$$
$$F_\theta = mr\ddot{\theta} + 2m\dot{r}\dot{\theta} \tag{13}$$

How can we write the laws of physics in a format that is invariant in any coordinate system? i.e. a form that is coordinate covariant, yet it can be used after we choose a coordinate system? This is what we will explore in what follows.

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Generalized coordinates

We already know from introductory physics the main quantities associated with the motion of a particle, namely position, velocity and acceleration. To define the position of a system of N particles in space we will use *N* radius vectors, i.e. *3N* coordinates. The number of independent quantities that need to be specified to uniquely determine the position of a system is called the number of degrees of freedom. This quantities (*3N*) do not need to be cartesian coordinates. We will use the coordinates that seem to be more convenient to the nature (or symmetry) of our problem. Any n quantities $q_1, q_2, ...q_n$ which completely define the position of a system with n degrees of freedom are called the generalized coordinates of the system, the derivatives $\dot{q}_1, \dot{q}_2, ...\dot{q}_n$ are called the generalized velocities.

It is clear that knowing all the generalized coordinates for a given system it is not enough to forecast its position at a later time. But observation indicates that if additionally we know all the generalized velocities at the same instant of time, we can calculate the subsequent evolution of the system. This means that knowing all the q_i and $\dot{q}_i$ at a given instant of time we can uniquely identified the accelerations $\ddot{q}_i$. The relations between the accelerations, velocities and coordinates are called the *equations of motion*. As we learned with Newton's equations they are second order differential equations for $q_i(t)$. Their integration makes it possible to determine the functions $q_i(t)$ and consequently the path or trajectory of the system.

The principle of least action

The principle of least action (sometimes called Hamilton's principle[2]) is the most general formulation of a universal law governing the motion of a mechanical system. The principle states that every mechanical system is characterized by a definite function $L(q_1, q_2, ...q_n, \dot{q}_1, \dot{q}_2, ...\dot{q}_n, t)$ or for the

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[2]Sir William Rowan Hamilton (3/4 August 1805 – 2 September 1865) was an Irish mathematician, astronomer, and physicist.
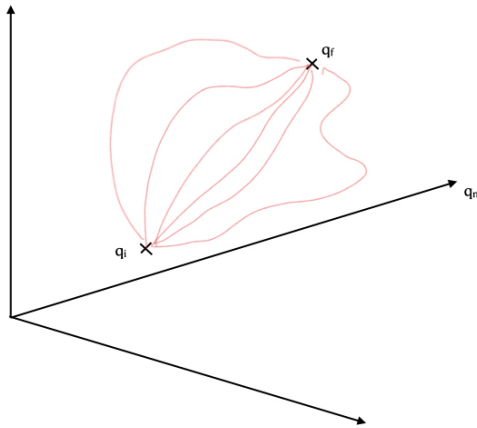
Figure 1: The infinitely many possible paths that the system could follow from q_i to q_f

sake of brevity $L(q, \dot{q}, t)$ which satisfies a very particular condition. The function L is called the Lagrangian of the system. If we assume that the system has at a given time t_i an initial position q_i and at a later time t_f a position q_f, we can construct the following integral

$$S = \int_{t_i}^{t_f} L(q, \dot{q}, t) dt \qquad (14)$$

which is called the action. We consider all paths between q_i and q_f in principle mechanically possible (see the illustration in figure 1).

The principle states that the action S given by the integral (14) takes the least possible value. Notice that L depends only on q and $\dot{q}$ reflecting our previous statement that knowledge of these quantities is sufficient to know the state of the system as a function of time.

Let's explore the implication of this principle. We see that it embodies the needed equations of motion that we are after.

Let's assume that $q = q(t)$ realizes the minimum of S. This of course means that whenever we consider

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$$q(t) + \delta q(t) \tag{15}$$

where $\delta q(t)$ is a very small quantity in the interval between t_i and t_f, S increases its value. $\delta q(t)$ is called a variation of $q(t)$. But notice that a requirement of our principle is that $\delta q_i(t) = \delta q_f(t) = 0$. What is the change in S when we change from $q(t)$ to $q(t) + \delta q(t)$?:

$$\delta S = \int_{t_i}^{t_f} L(q + \delta q(t), \dot{q} + \delta \dot{q}(t), t)dt - \int_{t_i}^{t_f} L(q, \dot{q}, t)dt \tag{16}$$

If we expand this expression in powers of q and $\dot{q}(t)$ the leading terms will be first order. Remember that we assume $\delta q(t)$ to be very small. So we can write the principle of least action as:

$$\delta S = \delta \int_{t_i}^{t_f} L(q, \dot{q}, t)dt = 0 \tag{17}$$

Performing the variation up to first order:

$$\int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = 0 \tag{18}$$

We can integrate by parts the second term:

$$\int_{t_i}^{t_f} \frac{\partial L}{\partial \dot{q}} \delta \dot{q} dt = \int_{t_i}^{t_f} \frac{\partial L}{\partial \dot{q}} \frac{d\delta q}{dt} dt = \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_i}^{t_f} - \int_{t_i}^{t_f} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \delta q dt = 0 \tag{19}$$

But $\delta q = 0$ so the first term in (19) is also 0. Using this result in (17) and (18) we get:

$$\delta S = \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q dt = 0 \tag{20}$$

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This means that the action will be a minimum (or more properly an extremum) if the following equations, called the Lagrange's equations (or sometimes the Euler-Lagrange equations), are satisfied:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{21}$$

Clearly is the system has more than one degree of freedom the equations are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, 2, ..., n) \tag{22}$$

If we know the Lagrangian of a physical system then equations (22) are the equations of motion of our system. The E-L equations are a set of n second order equations for the n unknown functions q_i. The general solution will have $2n$ arbitrary constants. To determine them and to consequently know the motion of the system we need to know the initial conditions which determine the state of the system at a given time. Typically we call these the initial positions and initial velocities.

Some properties of the Lagrangian

If we have a mechanical system which can be described as having two Lagrangians: L_A and L_B when we have them far apart enough that there is no interaction between both systems, the Lagrangian of the combined system is :

$$L = L_A + L_B \tag{23}$$

Also if we multiply a Lagrangian by a constant the equations of motion do not change. But this is equivalent to establishing the unit system to be used.

If two Lagrangians differ only by the total time derivative of a function $f(q, t)$,

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t) \tag{24}$$

and the action is

$$S' = \int_{t_i}^{t_f} L'(q, \dot{q}, t) dt = \int_{t_i}^{t_f} L(q, \dot{q}, t) dt + \int_{t_i}^{t_f} \frac{d}{dt} f(q, t) dt \tag{25}$$
$$= S + f(q_f, t_f) - f(q_i, t_i)$$

the two actions differ in a quantity that when variated it will give zero, i.e. $\delta S' = \delta S$. The implication of this result is that the Lagrangian is only defined up to an additive total time derivative of a function of the coordinates and time.

Galileo's relativity principle

We need frames to describe the mechanical evolution of physical systems or to analyze them. Galileo understood that there were privileged systems from where to describe motion. These systems of reference are infinitely many and equivalent differing between them in their relative constant velocity. A particular frame moving at the speed and in the same direction of an object with constant velocity will describe the motion of this object as at rest in the frame. These are the inertial frames or reference. In these frames space is homogeneous and isotropic. Notice that these statements are based on observations and experience. An important implicit concept here is locality. We know that the Earth is not an inertial system, and consequently the "local" universe does not appear isotropic or homogeneous. But local systems like the boat with the sailor dropping a cannonball from the mast and the observer on the pier do perceive their "local" universe as homogeneous and isotropic.

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Empowered with the concept of inertial systems we can infer the properties, or main features, or better put, the functional form of the Lagrangian for a free particle, i.e. a particle moving freely in an inertial system of reference. We conclude from the previous discussion that:

1. L can not depend on the position vector $\vec{r}$, be its length or the direction of its position, because otherwise a privileged direction would render other inertial systems of reference anisotropic or inhomogeneous.

2. L must be a function of the velocity $\vec{v}$ only but not of its direction.

3. L must then be a function of $|\vec{v}|$, ie. must be a function of its magnitude $\vec{v}^2 = v^2$.

4. Since L is independent of $\vec{r}$ we then have in (21) $\frac{\partial L}{\partial q} = \frac{\partial L}{\partial \vec{r}} = 0$.

5. The resulting E-L equations are $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) = 0$.

Hence

$$\frac{\partial L}{\partial \vec{v}} = constant \tag{26}$$

Having assume that our L is of the form $L = \alpha v^2$ this implies that

$$\vec{v} = constant. \tag{27}$$

In an inertial frame, any free motion takes place with a constant velocity (in magnitude and direction). This is Galileo's law of inertia.

A clear consequence of this formulation is that all inertial frames of reference will describe the motion of a free particle equivalently. Namely as a motion with constant velocity (in magnitude and direction). There are no absolute or preferred systems of reference. This is the Principle of Galilean Relativity.

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If two reference frames O and O' differ by a relative velocity $\vec{V}$, the coordinates of the free particle as described from the two frames are:

$$\vec{r} = \vec{r}' + \vec{V}t \tag{28}$$

where $\vec{r}$ are the coordinates of our particle from O and $\vec{r}'$ are the ones from O'. In our formalism we understand that the time

$$t = t'. \tag{29}$$

(28) and (29) constitute a Galilean transformation.

Galilean Relativity and Special Relativity

Galileo's relativity principle implies that the mechanical equations of motion are invariant under a Galilean transformation -(28) and (29)-. Notice that if we remove requirement (29) we need an equation relating the time between O and O'. When several experiments performed by the end of the XIX century were indicating a clear break in Galilean Relativity: the speed of light was the same for all inertial systems of reference, both (28) and (29) have to be replace to account for the observation. Galilean transformation were then replaced by the Lorentz transformations which respected the invariance of the speed of light. Notice the difference:

$$\vec{r}' = \frac{\vec{r} - \vec{V}t}{(1 - \frac{V^2}{c^2})^{1/2}} \tag{30}$$

$$t' = \frac{t - \frac{\vec{V}\cdot\vec{x}}{c^2}}{(1 - \frac{V^2}{c^2})^{1/2}} \tag{31}$$

with equations (28) and (29). Lorentz equations obey a different relativity, Special Relativity which states that the mechanical equations are the same in all inertial systems of reference with the addition that the velocity of light c is the same in all of them. Notice that if $\vec{V} \ll c$ equations (30) and (31) become (28) and (29).

The Lagrangian of a free particle

To figure out the functional form of the Lagrangian for a free particle we could use our knowledge from the previous section. If in system O the particle has velocity $\vec{v}$, in O' it should have velocity $\vec{v}' = \vec{v} + \vec{\epsilon}$. So the Lagrangian L' can only differ from $L(v^2)$ if at all by a total derivative of

a function of the coordinates and time (remember the discussion after the end of section **3.1**).

We then have

$$L(v'^2) = L\left((\vec{v} + \vec{\epsilon})^2\right) = L(v^2) + \frac{\partial L}{\partial v^2} 2\vec{v} \cdot \vec{\epsilon}. \tag{32}$$

The second term in the previous equation is a total derivative only if it is a linear function of $\vec{v}$. This implies that $\frac{\partial L}{\partial v^2}$ does not depend on $\vec{v}$. This implies then that the Lagrangian has to be proportional to v^2:

$$L = \frac{1}{2}\alpha v^2 \tag{33}$$

This Lagrangian satisfies the principle of relativity. If we replace $\vec{\epsilon}$ for a finite relative velocity $\vec{V}$ between frames O and O'.the Lagrangian in the prime system will be

$$L' = \frac{1}{2}\alpha v'^2 = \frac{1}{2}\alpha(\vec{v} + \vec{V})^2 = \frac{1}{2}\alpha v^2 + \alpha\vec{v} \cdot \vec{V} + \frac{1}{2}\alpha V^2 \tag{34}$$

$$L' = L + \frac{d}{dt}(\alpha\vec{r} \cdot \vec{V} + \frac{1}{2}\alpha V^2 t) \tag{35}$$

α is the mass of the particle so we will call it m:

$$L = \frac{1}{2}mv^2 \tag{36}$$

and in the case of a system of n particles which did not interact we have:

$$L = \sum_{i=1}^{n} \frac{1}{2}m_i v_i^2 \tag{37}$$

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in section (3.1) we said that if we multiply a Lagrangian by a constant the equations of motion do not change and that the multiplying factor is equivalent to establishing the unit system to be used.

We can also see that $m > 0$. The principle of least action establishes that

$$S = \int_1^2 \frac{1}{2}mv^2 dt \tag{38}$$

has a minimum from point 1 to point 2. That would be impossible if the integrand is definite negative.

The Lagrangian of a free particle in different coordinates

To obtain the Lagrangian in different coordinates it suffices to notice that:

$$v^2 = \left(\frac{dl}{dt}\right)^2 = \frac{(dl)^2}{(dt)^2} \tag{39}$$

In cartesian coordinates then $dl^2 = dx^2 + dy^2 + dz^2$ and:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{40}$$

In cylindrical coordinates $dl^2 = dr^2 + r^2 d\phi^2 + dz^2$ and then

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) \tag{41}$$

In spherical coordinates $dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ and then:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta) \tag{42}$$

The Lagrangian for a system of particles

Let's consider a system of particles which may interact which each other but with no other bodies. This is a closed system. In this situation, experience from observations indicate that the interaction between the particles can be represented by a function of the coordinates (i.e. the position of the particles), $U(\vec{r}_1, \vec{r}_2, ...\vec{r}_n)$. Of course U reflects the nature of the interaction at play. Notice that we use U with a $-$ sign in the Lagrangian.

$$L = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 - U(\vec{r}_1, \vec{r}_2, ..., \vec{r}_n) \tag{43}$$

where $\vec{r}_i$ is the radius vector of the i particle. This is the typical form for the Lagrangian describing a closed system. $\sum_{i=1}^{n} \frac{1}{2} m_i v_i^2$ is called the kinetic energy of the system and U its potential energy. We will discuss in the next lesson the reason for choosing these names for these quantities.

A comment about the function U. We are taking a phenomenological approach to describe the interactions between particles. We are assuming that U is a function of the coordinates of the particles. If they change their position then U changes. Notice that in our description we are not assuming a potential function that depends explicitly on the time (although it does reflect evolution with time, through the kinetic energy U is only a function of the coordinates). This implies an instantaneous propagation of the interactions (action at distance). If we were to accept a time that is frame dependent like with Lorentz transformations in Special Relativity, we would need an entire reformulation of our theory. So far we are working assuming (28) and (29). If we were not we would need a relativistic theory of mechanics. We may say something about it at the end of this course.

Regarding the isotropy of time, which we mentioned before, we can see it clearly expressed in the Lagrangian where the kinetic energy shows implicitly its time dependence. If we substitute $-t$ for t clearly nothing changes. The equations of motion remain the same. It is in this sense that the laws of classical mechanics are reversible. The equations of motion

given the Lagrangian are

$$\frac{d}{dt}\frac{\partial L}{\partial \vec{v}_i} = \frac{\partial L}{\partial \vec{r}_i} \qquad (44)$$

If we use the Lagrangian from (43) in (44) we get

$$m_i \frac{d}{dt}\vec{v}_i = -\frac{\partial U}{\partial \vec{r}_i} \qquad (45)$$

These are called Newton's equations of motion which are the foundation of the mechanics of particles. we called force the vector:

$$\vec{F} = -\frac{\partial U}{\partial \vec{r}_i} \qquad (46)$$

which is acting on particle i. Similarly to U it depends on the coordinates of the particles, but not on their velocities. This is showing that the acceleration vectors are only functions of the coordinates.

Galileo and the definition of acceleration

Galileo was the first scientist to define acceleration as the time rate change of velocity: a crucial concept. But before this successful definition, his first attempts were defining it as the change of velocity with distance. Let's explore these ideas. Such a definition would lead to the trajectories being exponentials, not parabolas.

$$a = \frac{dv}{dx} = g \tag{47}$$

where $g = -9.8m/s^2$ From there you get

$$dv = gdx \rightarrow \int dv = \int gdx \rightarrow v = gx + const. \tag{48}$$

But this is

$$\frac{dx}{dt} = gx + const. \tag{49}$$

And from there we get:

$$\frac{dx}{x} = gdt \rightarrow \int \frac{dx}{x} = \int gdt \rightarrow \ln x = gt \tag{50}$$

where $\ln$ is the natural logarithm. We can take the exponential of both sides to get:

$$x(t) = \exp(gt) = e^{-9.8t} \tag{51}$$

And of course these curves are very different from parabolas. It is interesting to plot the two solutions that could be obtained with a simple case. Let's look at a stone drop vertically from a 10 meters height.
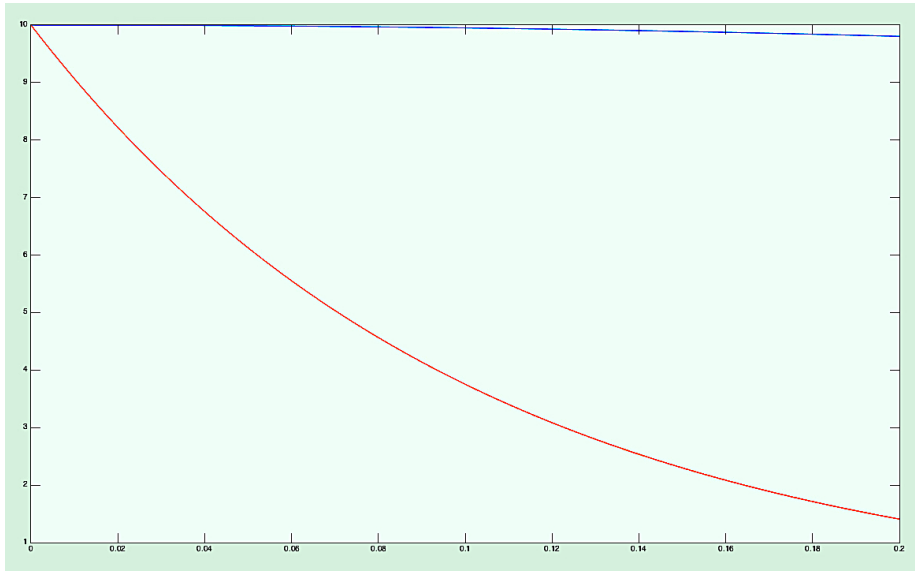
Figure 2: An object dropped from 10m above the ground falling exponentially -blue- and parabolically -red-

The two solutions are:

$$x(t) = 10 \exp(gt) = 10e^{-9.8t} \tag{52}$$

and

$$x(t) = 10 - \frac{9.8}{2}t^2 \tag{53}$$

The exponential curve in Figure 2, does not correspond to the actual trajectory. Experiments determine clearly that the parabola is the right mathematical expression, This is a conclusive proof that defining acceleration as the change in velocity with time is the right one.

Notice that U can be defined up to constant. It is natural to make $U = 0$ when the separation between particles is large enough that no interaction

is experienced. When using coordinates other than Cartesian ones we will use the following transformation:

$$x_i = f_i(q_1, q_2, ..., q_n), \quad \dot{x}_i = \sum_{k=1}^{n} \frac{\partial f_i}{\partial q_k} \dot{q}_k \tag{54}$$

We substitute them in

$$L = \frac{1}{2} \sum_{i=1}^{n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - U \tag{55}$$

to get the following Lagrangian

$$L = \frac{1}{2} \sum_{i,k} a_{ik}(q)\dot{q}_i\dot{q}_k - U(q), \tag{56}$$

We notice that in the new coordinates the Lagrangian will be still a quadratic function of the generalized velocities but the kinetic energy may depend on the new coordinates as well.

The Lagrangian of open systems

If we have an open system A which interacts with a system B , we can consider $A + B$ as a closed system. We can find the Lagrangian of A by using the Lagrangian L_{A+B} and replace the coordinates q_B by functions of time. If $A + B$ is closed we get then $L = T_A(q_A, \dot{q}_A) + T_B(q_B, \dot{q}_B) - U(q_A, q_B)$, where the first two terms are the kinetic energies of the system A and B and the third term is their combined potential energy. We can now substitute for q_B the given functions of time and remove $T_B(q_B, \dot{q}_B)$ which, depending on time, is a total time derivative of a function of time. We will

get $L = T_A(q_A, \dot{q}_A) - U(q_A, q_B(t))$.

The motion of a system in an external field is described by a Lagrangian where the potential energy may be an explicit function of time.

As an example the Lagrangian for a single particle moving in an external field has this general form

$$L = \frac{1}{2}mv^2 - U(\vec{r}, t) \tag{57}$$

The corresponding equation of motion is

$$m\dot{\vec{v}} = -\frac{\partial U}{\partial \vec{r}} \tag{58}$$

Uniform field

When we consider physical systems where the same force acts on a particle at any point in the field considered, the field is called uniform. The associated potential energy is then

$$U = -\vec{F} \cdot \vec{r}. \tag{59}$$

Application of the Lagrangian formalism to physical problems

The actual number of degrees of freedom of a physical system can be further reduced by constraints. i.e. a system subjected to move within a confined region of space. If the friction and masses associated with the structures imposing these constraints can be neglected we can follow the formalism described in this chapter to find the equations of motion. We will just use as many generalized coordinates as the actual number of degrees

of freedom of our system. i.e. if we impose l number of constraints on a system with originally n degrees of freedom then we can find the equations of motion of our system utilizing $n - l$ generalized coordinates.

Generalized forces and Lagrange multipliers

We can consider the case where the n generalized coordinates are not independent. All the virtual displacements δq_i are no longer possible but subjected to the imposed constraints. We can consider just one condition

$$\sum_{i=1}^{n} \Lambda_i \delta q_i = 0 \tag{60}$$

Notice that when given the constraints this equation defines Λ_i. There is a very special kind of constraint, the one for which a quantity Φ can be define such

$$\sum_{i=1}^{n} \Lambda_i \delta q_i = d\Phi(q) \tag{61}$$

In this case the constraint is equivalent to $\Phi(q)$ being a constant and it is called holonomic. In this case we can express one generalized coordinate as a function of the $n - 1$ others. If the constraint is non-holonomic then we need to keep all the coordinates and use Lagrange multipliers. In that case if there are s differential constraints of the form (60) we have

$$A_i(q, \dot{q}, \ddot{q}, t) = Q_i(q, \dot{q}, t) + \sum_{k=1}^{s} \lambda_k \Lambda_i{}^k(q, t) \tag{62}$$
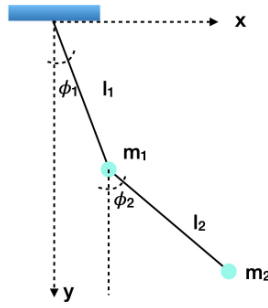
Examples

Problem 1

Figure 3: A double pendulum

Find the Lagrangian for the double pendulum (see Figure 3) placed near the surface of the Earth (acceleration g).

Solution

The coordinates are ϕ_1 and ϕ_2. Then

$$T_1 = \frac{1}{2}m_1 l_1^2 \dot{\phi}_1^2, \qquad U_1 = -m_1 g l_1 \cos\phi_1 \tag{63}$$

To find T for the second particle we start with Cartesian coordinates: We have

$$x_2 = l_1 \sin\phi_1 + l_2 \sin\phi_2, \qquad y_2 = l_1 \cos\phi_1 + l_2 \cos\phi_2; \tag{64}$$

After taking the derivative and squaring it we get

$$T_2 = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right)$$
$$= \frac{1}{2}\left(l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 \cos(\phi_1 - \phi_2)\dot{\phi}_1 \dot{\phi}_2\right) \tag{65}$$

where we have used that $\cos(\phi_1 - \phi_2) = \cos\phi_1 \cos\phi_2 - \sin\phi_1 \sin\phi_2$. The Lagrangian then is

$$L = \frac{1}{2}\left(l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2l_1 l_2 \cos(\phi_1 - \phi_2)\dot{\phi}_1 \dot{\phi}_2\right) + (m_1 + m_2)g l_1 \cos\phi_1 + m g l_2 \cos\phi_2 \tag{66}$$

Figure 4: A sling

A rigid rod rotates (see Figure 4) in a horizontal plane around a fixed point O with angular speed $\omega = \dot{\phi}$. A bead of mass m can slide on it without friction. Treat it like a point particle and find its Lagrangian and the equations of motion.

Solution

We pick the plane of motion to be $x - y$ with the z axis perpendicular to it. This is a Galilean system in the sense that it is fixed and not rotating with the rod. We will use polar coordinates. Let ϕ be the angle and r the radius indicating the position of the bead.

The kinetic energy is

$$T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2). \tag{67}$$

Notice that there is no potential energy, but this is not a free particle either. The speed $\dot{\phi}$ is provided by an imposed external ω (i.e. an electric motor). The weight of the bead due to gravity is balanced by the normal reaction force of the rod, but there is no friction. Consequently the external forces

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add up to 0. This is still a constrained system. So the Lagrangian is

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2). \tag{68}$$

And the E-L equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{69}$$

How about ϕ? ϕ is constrained by the fact that $\dot{\phi} = \omega$. So the system has only one degree of freedom. And the equation of motion is:

$$\ddot{r} - r\dot{\phi}^2 = 0 \tag{70}$$

The solution of (70) is

$$r(t) = A\cosh(\omega t) + B\sinh(\omega t) \tag{71}$$

Our initial conditions could be $\dot{r}(0) = 0$ and $r(0) = r_0$. In that case the solution is

$$r(t) = r_0 \cosh(\omega t) \tag{72}$$

Note: (70) express the fact that the centripetal acceleration $\ddot{r}$ is equal to a "centrifugal force" $r\dot{\phi}^2$.