Introduction to Astrophysics
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High Energy Bound-Free Absorption

If the photon is absorbed by a bound electron with a binding energy much less than the photon energy, there is essentially no difference with the previous case of free-free absorption. As a result the temperature dependence is also with a factor $\beta = -7/2$. What is different though is that the density of these electrons is not the ambient density of free electrons, but an average square of the bound electron wave function, so the absorption rate $c\rho\kappa$ is proportional just to the density of atoms, and hence $\alpha = 0$ rather than $\alpha = 1$. The contribution to opacity of this sort of photon absorption is often grouped with free–free absorption in what is called *Kramers opacity*.

Bound–Bound Absorption and Low-Energy Bound–Free Absorption

In these cases the photon is absorbed by a bound electron whose binding energy is at least comparable to the photon energy. This contribution to opacity involves complications of atomic physics not present for other contributions which Weinberg does not discuss in his book. The heating of interstellar hydrogen by low-energy bound–free absorption of photons from hot stars will be covered later.

We will consider now:

Nuclear Energy Generation

Primer on atomic nomenclature

The atomic number or proton number (symbol Z) of a chemical element is the number of protons found in the nucleus of every atom of that element. It uniquely identifies a chemical element and it is identical to the charge of the nucleus. In an uncharged atom, the atomic number is also equal to the number of electrons.

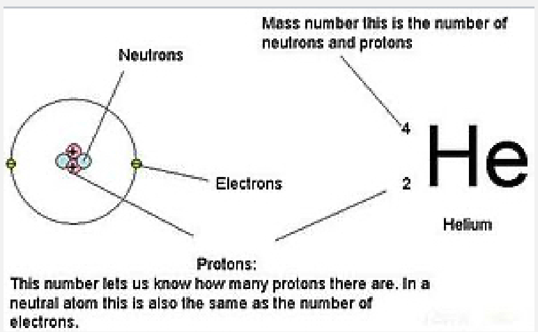

Figure 1: Atomic number notation. ©Wikipedia

The sum of the atomic number Z and the number of neutrons N gives the mass number A of an atom. Since protons and neutrons have approximately the same mass (and the mass of the electrons is negligible for many purposes) and the mass defect of nucleon binding is also small compared to the nucleon mass, the atomic mass of any atom, when expressed in unified atomic mass units is within 1% of the whole number A.

Atoms with the same atomic number but different neutron numbers, and hence different mass numbers, are known as isotopes. A little more than three-quarters of naturally occurring elements exist as a mixture of isotopes, and the average isotopic mass of an isotopic mixture for an element, determines the element's standard atomic weight. **Important note: the convention when naming isotopes (and not using chemical symbols, as i.e. ^{238}U) to either type the element's name and then put the mass number after the name or just U^{238}**

Nucleosynthesis

(From wikipedia)

Nucleosynthesis is the process that creates new atomic nuclei from pre-existing nucleons (protons and neutrons) and nuclei. The first nuclei were formed a few minutes after the Big Bang, through nuclear reactions in a process called Big Bang nucleosynthesis. After about 20 minutes, the universe had expanded and cooled to a point at which these high-energy collisions among nucleons ended, so only the fastest and simplest reactions occurred, leaving our universe containing about 75% hydrogen and 24% helium by mass. The rest is traces of other elements such as lithium and the hydrogen isotope deuterium. Nucleosynthesis in stars and their explosions later produced the variety of elements and isotopes that we have today, in a process called cosmic chemical evolution. The amounts of total mass in elements heavier than hydrogen and helium (called "metals" by astrophysicists) remains small (few percent), so that the universe still has approximately the same composition.

Stars fuse light elements to heavier ones in their cores, giving off energy in the process known as stellar nucleosynthesis. Nuclear fusion reactions create many of the lighter elements, up to and including iron and nickel in the most massive stars. Products of stellar nucleosynthesis remain trapped in stellar cores and remnants except if ejected through stellar winds and explosions. The neutron capture reactions of the r-process and s-process create heavier elements, from iron upwards.

Supernova nucleosynthesis within exploding stars is largely responsible for the elements between oxygen and rubidium: from the ejection of elements produced during stellar nucleosynthesis; through explosive nucleosynthesis during the supernova explosion; and from the r-process (absorption of multiple neutrons) during the explosion. Neutron star mergers are a recently discovered major source of elements produced in the r-process. When two neutron stars collide, a significant amount of neutron-rich matter may be ejected which then quickly forms heavy elements.

Cosmic ray spallation is a process wherein cosmic rays impact nuclei and fragment them. It is a significant source of the lighter nuclei, particularly ^{3}He, ^{9}Be and ^{10,11}B, that are not created by stellar nucleosynthesis. Cosmic ray spallation can occur in the interstellar medium, on asteroids and meteoroids, or on Earth in the atmosphere or in the ground. This contributes to the presence on Earth of cosmogenic nuclides.

On Earth new nuclei are also produced by radiogenesis, the decay of long-lived, primordial radionuclides such as uranium, thorium, and potassium-40.

Remember that we call the nuclear energy production per mass $\epsilon(\rho, T)$. What we want to do is to estimate the exponents of $\epsilon(\rho, T)$ when we adjust the function representing this production rate using a power-law expression:

$$\epsilon(\rho, T) \simeq \epsilon_1 \rho^\lambda (k_B T)^\nu, \tag{148}$$

where ϵ_1, λ and ν are independent of ρ and T.

As we saw in the colored box the nuclear material left over from the first three minutes of the big bang was chiefly ^{1}H (that is, protons), plus about 25% by mass ^{4}He, and only a trace of ^{2}H, ^{3}He, and ^{7}Li. These light nuclei have less binding energy per nucleon than nuclei of medium atomic weight like iron and nickel, so energy can be gained by fusion of hydrogen and helium into heavier elements. But there are no stable nuclei with five or eight nucleons, so it is difficult (though not impossible) to gain energy from helium in ^{1}H-^{4}He or ^{4}He-^{4}He collisions.

As long as hydrogen lasts in the center of a star, the dominant source of nuclear energy will be the fusion of ^{1}H into ^{4}He, which has by far the greatest binding energy of any of these light elements. There are two chief routes by which hydrogen can fuse into helium. One is the proton–proton chain of which the simplest version is

$$\begin{aligned} \mathrm{I}: &\quad {}^1\mathrm{H} + {}^1\mathrm{H} \rightarrow {}^2\mathrm{H} + \mathrm{e}^+ + \nu_\mathrm{e} + 1.18\mathrm{MeV} \\ \mathrm{II}: &\quad {}^1\mathrm{H} + {}^2\mathrm{H} \rightarrow {}^3\mathrm{He} + \gamma + 5.49\mathrm{MeV} \\ \mathrm{III}: &\quad {}^3\mathrm{He} + {}^3\mathrm{He} \rightarrow {}^4\mathrm{He} + {}^1\mathrm{H} + {}^1\mathrm{H} + 12.85\mathrm{MeV} \end{aligned} \tag{149}$$

The energies listed here for the proton–proton chain and below for the CNO cycle are the energies for each reaction actually deposited in the stellar material. Thus, where a positron is emitted, these energies include not only the rest energy $m_e c^2$ of the emitted positron e^+ but also the rest energy of the electron with which that positron inevitably annihilates. On the other hand, the mean energy of the accompanying neutrino ν_e is subtracted from the energy released, since virtually all neutrinos leave the star.

Each step of the PP I chain depicted above has its own reaction rate, since different Coulomb barriers and cross sections are involved. The slowest step in the sequence is the initial one, because it involves the decay of a proton into a neutron via $p^+ \rightarrow n + e^+ + \nu_e$. Such a decay involves the weak force, another of the four known forces.

Notice that there are other branches for the PP chain, namely, the so-called PP II chain,

$$
\begin{aligned}
&^3\mathrm{He} +^4 \mathrm{He} \rightarrow^7 \mathrm{Be} + \gamma \\
&^7\mathrm{Be} + \mathrm{e}^- \rightarrow^7 \mathrm{Li} + \nu_\mathrm{e} \\
&^7\mathrm{Li} +^1 \mathrm{H} \rightarrow^4 \mathrm{He} +^4 \mathrm{He}
\end{aligned}
$$

and the PP III chain,

$$
\begin{aligned}
&^7\mathrm{Be} +^1 \mathrm{H} \rightarrow^8 \mathrm{B} + \gamma \\
&^8\mathrm{B} \rightarrow^8 \mathrm{Be} + \mathrm{e}^+ + \nu_\mathrm{e} \\
&^8\mathrm{Be} \rightarrow^4 \mathrm{He} +^4 \mathrm{He}
\end{aligned}
$$

The branching order and their ratios are depicted in the figure **??**.

The other route is the CNO cycle, which in its simplest variant is

$$
\begin{aligned}
\mathrm{I}: &\quad ^1\mathrm{H} +^{12} \mathrm{C} \rightarrow^{13} \mathrm{N} + \gamma + 1.95\mathrm{MeV} \\
\mathrm{II}: &\quad ^{13}\mathrm{N} \rightarrow^{13} \mathrm{C} + \mathrm{e}^+ + \nu_\mathrm{e} + 1.50\mathrm{MeV} \\
\mathrm{III}: &\quad ^1\mathrm{H} +^{13} \mathrm{C} \rightarrow^{14} \mathrm{N} + \gamma + 7.54\mathrm{MeV} \\
\mathrm{IV}: &\quad ^1\mathrm{H} +^{14} \mathrm{N} \rightarrow^{15} \mathrm{O} + \gamma + 7.35\mathrm{MeV} \\
\mathrm{V}: &\quad ^{15}\mathrm{O} \rightarrow^{15} \mathrm{N} + \mathrm{e}^+ \nu_\mathrm{e} + 1.73\mathrm{MeV} \\
\mathrm{VI}: &\quad ^1\mathrm{H} +^{15} \mathrm{N} \rightarrow^{12} \mathrm{C} +^4 \mathrm{He} + 4.96\mathrm{MeV}
\end{aligned}
\tag{150}
$$

where carbon, nitrogen, and oxygen nuclei are present in the interstellar matter from which stars like the Sun are formed (i.e. left over from nuclear processes in an earlier generation of stars). They are catalysts, neither created nor destroyed in a complete cycle. In both cases there are side branches and extensions with different rates involved, but these simple versions provide us with sufficient examples to illustrate how $\epsilon(\rho, T)$ is estimated. We will not go into the detailed calculation of the rates of these various nuclear reactions. We are interested in identifying various suppression factors in the rates that tell us a good deal about which reactions are dominant, and

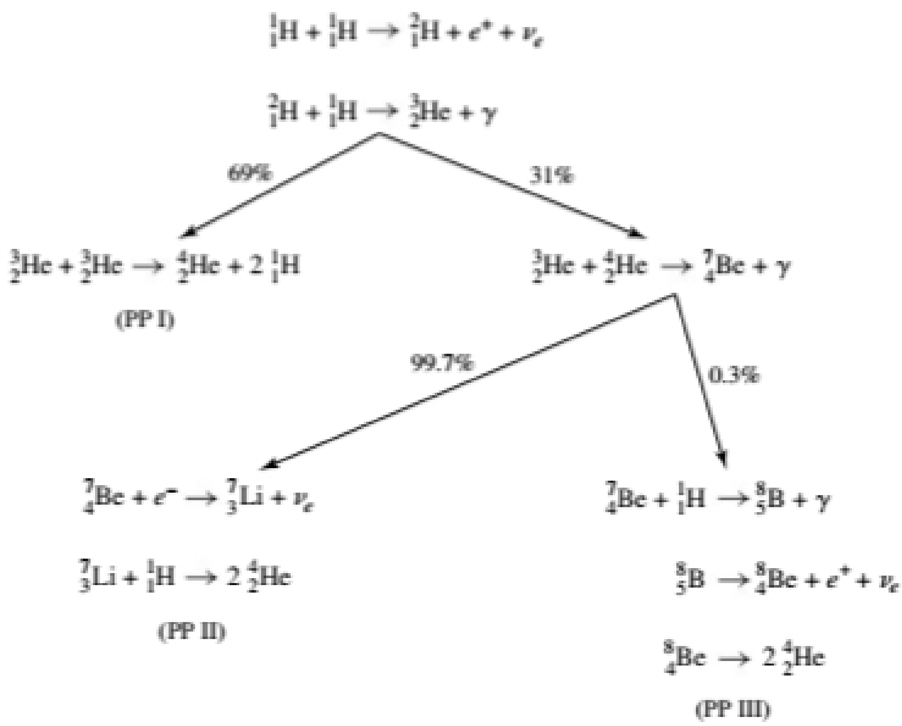

Figure 2: The three branches of the pp-chain and the ratios observed in the Sun. From Carroll&Ostlie

about their temperature dependence.

Electromagnetic coupling

The rate of any reaction in which a single photon is emitted (such as $^1\mathrm{H} +^2 \mathrm{H} \rightarrow^3 \mathrm{He} + \gamma$ in the proton-proton cycle or $^1\mathrm{H} +^{12} \mathrm{C} \rightarrow^{13} \mathrm{N} + \gamma$ in the CNO cycle) is suppressed by a factor of order $e2/\hbar c = 1/137$.

In physics, the fine-structure constant, also known as Sommerfeld's constant, commonly denoted by α, is a fundamental physical constant which quantifies the strength of the electromagnetic interaction between elementary charged particles. It is a dimensionless quantity related to the elementary charge e, quantifying the strength of the coupling of an elementary charged particle with the electromagnetic field, by the formula $4\pi\epsilon_0\hbar c\alpha = e^2$. As a dimensionless quantity, its numerical value, is $\alpha = e^2/\hbar c = 1/137$ is independent of the system of units used.

While there are multiple physical interpretations for α, it received its name from Arnold Sommerfeld, who introduced it in 1916, when extending the Bohr model of the atom.It quantifies the gap in the fine structure of the spectral lines of the hydrogen atom, which had been measured precisely by Michelson and Morley in 1887. The relative strengths of the coupling constants are below

Strong	α_s	1
E-M	α	1/137
Weak	α_w	10^{-6}
Gravity	α_g	10^{-39}

Weak coupling

The rate of any reaction in which a proton turns into a neutron with the emission of a positron and neutrino (such as the first step $^1\mathrm{H} +^1 \mathrm{H} \rightarrow 2\mathrm{H} + \mathrm{e}^+ + \nu_\mathrm{e}$ in the proton–proton cycle or the beta decays of $^{13}\mathrm{N}$ and $^{15}\mathrm{O}$ in the CNO cycle) is suppressed by two factors of the weak coupling constant $G_{wk} = 1.1664 \times 10^{-11}\ \mathrm{MeV}^{-2}$. Since the typical energy involved in these nuclear reactions is about 1 MeV, weak interaction processes are typically suppressed by a dimensionless factor of order 10^{-22}.

Coulomb barrier

The temperature dependence of nuclear reaction rates is chiefly due to the necessity for colliding nuclei to leak through the Coulomb barrier, the field of electrostatic repulsion between positively charged atomic nuclei. The calculation of the effect of the Coulomb barrier on reaction rates requires the use of elementary quantum mechanics. The result is that a reaction involving two nuclei of atomic numbers Z_1 and Z_2 and an energy of relative motion E is suppressed by a factor of order

$$B(E) = exp\left[-\pi \mathrm{Z}_1\mathrm{Z}_2 e^2\sqrt{\frac{2\mu}{\hbar^2 E}}\right] = \exp\left(\frac{C}{\sqrt{E}}\right), \qquad (151)$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass and C is the constant in the exponent in Eq. (151):

$$C = \pi \mathrm{Z}_1\mathrm{Z}_2 e^2\sqrt{\frac{2\mu}{\hbar^2}}, \qquad (152)$$

The nuclei colliding in a star have a range of values for the energy E of relative motion, with probabilities governed by the requirements of kinetic theory at temperature T. Assuming that nuclei spend most of their time sufficiently far from other nuclei that their energy is mostly kinetic, the probability of finding a pair of nuclei in a range of momenta $d^3p_1 d^3p_2$ is proportional to

$$\exp\left(-\frac{(\vec{p}_1)^2}{2m_1k_BT}-\frac{(\vec{p}_2)^2}{2m_2k_BT}\right)d^3p_1d^3p_2=\exp\left(-\frac{E}{k_BT}\right)d^3p \\ \times\exp\left(-\frac{(\vec{P})^2}{2(m_1+m_2)k_BT}\right)d^3P \tag{153}$$

where $\vec{P}\equiv\vec{p}_1+\vec{p}_2$ is the total momentum, and $E=p^2/2\mu$ is the energy of relative motion, with $\vec{p}\equiv\mu(\vec{p}_1/m_1-\vec{p}_2/m_2)$ the relative momentum. The rate ϵ of nuclear reactions per gram is then of the form

$$\begin{aligned}\epsilon(\rho,T)&=\int_0^\infty dEf(E,\rho,T)\exp(-E/k_BT)B(E)\\ &=\int_0^\infty dEf(E,\rho,T)\exp\left(-\frac{E}{k_BT}-\frac{C}{\sqrt{E}}\right).\end{aligned} \tag{154}$$

where $f(E,\rho,T)$ arises from power-law factors in the thermal distribution of E and P and in the probability of the nuclear reaction occurring when the nuclei reach zero separation, In practice, k_BT is always much less than C^2, so the exponential $\exp$ will be very small unless E is much greater than k_BT, in which case $\exp(-E/k_BT)$ will be very small. The exponential in Eq. (153) is therefore very sharply peaked at the energy E_T where a maximum occurs for $\left(-\frac{E}{k_BT}-\frac{C}{\sqrt{E}}\right)$. To find E_T then

$$\frac{d}{dE}\left(-\frac{E}{k_BT}-\frac{C}{\sqrt{E}}\right)=-\frac{1}{k_BT}+\frac{C}{2E_T^{3/2}}=0 \tag{155}$$

which gives

$$E_T=\left(\frac{Ck_BT}{2}\right)^{2/3} \tag{156}$$

The dominant factor in the temperature dependence of the reaction rate (154) is the exponential function evaluated at $E = E_T$

$$B(E_T) = \exp\left(-\frac{E_T}{k_B T} - \frac{C}{\sqrt{E_T}}\right) = \exp\left(-3\left(\frac{\pi Z_1 Z_2 e^2 \sqrt{\mu}}{\hbar\sqrt{2k_B T}}\right)^{2/3}\right). \tag{157}$$

which numerically is:

$$B(E_T) = \exp\left[-\left(Z_1^2 Z_2^2 (\mu/m_p) \times \frac{7.726 \times 10^{10}\mathrm{K}}{T}\right)^{1/3}\right]. \tag{158}$$

where m_P is the proton mass.

The barrier penetration factor, $B(E)$ is the main factor determining the temperature dependence of the reaction rates. We can then use the above estimate of the Coulomb barrier to determine the exponent ν in the formula that is used to estimate the temperature dependence of the energy generation rate $\epsilon(\rho, T) \simeq \epsilon_1 \rho^\lambda (k_B T)^\nu$. We take thinking that $B_T(T) = T^\nu$

$$\nu = T\frac{d}{dT}B_T \simeq \frac{1}{3}\left(Z_1^2 Z_2^2 (\mu/m_p) \times \frac{7.726 \times 10^{10}\mathrm{K}}{T}\right)^{1/3}. \tag{159}$$

(This $T^{-1/3}$ temperature dependence is sufficiently weak to justify approximating ϵ as proportional to a constant power of temperature.) Eqs. (158) and (159) indicate as general rule that ν is one-third the absolute value of the exponent in whatever barrier penetration factor dominates the temperature dependence of the energy generation rate. Let us now apply these general remarks to stars that derive their nuclear energy either from the proton–proton chain or from the CNO cycle.