Introduction to Astrophysics
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If we look at equations (6) (7) (101) and (107)

$$\frac{dP(r)}{dr} = -G\frac{\rho(r)\mathcal{M}(r)}{r^2} \tag{6}$$

$$\frac{d\mathcal{M}(r)}{dr} = 4\pi r^2 \rho(r) \tag{7}$$

$$\frac{d\mathcal{L}(r)}{dr} = 4\pi r^2 \epsilon(r)\rho(r), \tag{101}$$

$$\frac{dT(r)}{dr} = -\frac{3\kappa(r)\rho(r)\mathcal{L}(r)}{4acT^3(r)4\pi r^2}. \tag{107}$$

The pressure in (6) is actually

$$P = p_{gas} + p_{rad} \tag{120}$$

where the pressure due to radiation is given by

$$p_{rad} = \frac{a}{3}T^4 \tag{121}$$

At this point it is relevant to notice that the gas pressure diminishes as the radio increases (remember that from basic thermodynamics $p_{gas} = \frac{\rho k_B T}{M}$) As we move towards the surface not only the density, but also the temperature decrease. And then $dp_{gas}/dr < 0$ Taking the derivatives of both sides in eq (121) respect to the radius and using the result for dT/dr in (107) we get

$$\frac{dp_{rad}(r)}{dr} = -\frac{\kappa(r)\rho(r)\mathcal{L}(r)}{4\pi c r^2}. \tag{122}$$

and subtracting from (6) we get

$$-\frac{\kappa(r)\rho(r)\mathcal{L}(r)}{4\pi cr^2}+\frac{G\mathcal{M}(r)\rho(r)}{r^2}=-\frac{dp_{gas}}{dr}>0. \tag{123}$$

which means that everywhere in the star

$$\kappa(r)\mathcal{L}(r)<4\pi Gc\mathcal{M}(r).$$

Evaluating the inequality at the star radius where $\mathcal{L}(r=R)=L$ and $\mathcal{M}(r=R)=M$ we get

$$\kappa(R)L<4\pi GcM \tag{124}$$

If this inequality were broken, the outer layers of the star would be blown off due to the radiation pressure alone. When the opacity in the star's outer layers is due to Thomson scattering the inequality (124) is known as the **Eddington limit**. This inequality also limits the luminosity that can be produced by spherically symmetric accretion onto a star or galactic nucleus.

Notice that if the gas pressure is negligible compared with the radiation pressure (which only happens in very massive stars) we get from $dp_{gas}/dr\sim 0$

$$\kappa(R)L=4\pi GcM \tag{125}$$

Opacity

Equation (107)

$$\frac{dT(r)}{dr}=-\frac{3\kappa(r)\rho(r)\mathcal{L}(r)}{4acT^3(r)4\pi r^2}. \tag{107}$$

involves the opacity κ which in general is given

$$\kappa \equiv \kappa_{abs} + \kappa_{out} - \kappa_{in} \tag{126}$$

where the κ in (107) is given by the Rosseland mean

$$\int d\nu \frac{1}{\kappa(\rho, T, \nu)} \left(\frac{\partial B(\nu, T)}{\partial T} \right) = \frac{4aT^3}{\kappa(\rho, T)}$$

and where $B(\nu, T)$ is given by

$$B(\nu, T(\mathbf{x})) = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/k_B T} - 1} \tag{127}$$

The first term in Eq. (126) is defined so that $c\rho\kappa_{abs}$ is the net rate of absorption – the average rate per photon at which photons are absorbed, less the rate per initial photon at which photons with the same momentum are created by stimulated emission. If Γ_{abs} is the rate of absorption alone, then when stimulated emission is taken into account, the net rate of photon absorption is

$$c\rho\kappa_{abs}(\rho, T, \nu) = \Gamma_{abs}(\rho, T, \nu) \left[1 - e^{-h\nu/k_B T} \right] \tag{128}$$

The second and third terms in Eq. (126) are defined so that $c\rho\kappa_{out}$ and $c\rho\kappa_{in}$ are the rates at which photons are scattered out of or into any given direction. In cases where scattering occurs in a collision with a single particle, such as an electron or atom, these terms are given by Eqs. (72):

$$\int d^2\hat{n}' \kappa_s(\hat{n} \to \hat{n}'; \mathbf{x}, \nu) \equiv \kappa_{out}(\mathbf{x}, \nu) \tag{72}$$

from where we define

$$\kappa_{out} = N_{scat} \int d^2\hat{n}' \sigma_{scat}(\hat{n} \to \hat{n}'), \tag{129}$$

and (78)

$$\begin{aligned} \hat{n}'_i \cdot \hat{n}' \kappa_{in}(\mathbf{x}, \nu) = \kappa_{in}(\mathbf{x}, \nu) &= \int d^2\hat{n}(\hat{n} \cdot \hat{n}')\kappa_s(\hat{n}' \to \hat{n}; \mathbf{x}, \nu) \\ &= \int d^2\hat{n}(\hat{n} \cdot \hat{n}')\kappa_s(\hat{n} \to \hat{n}'; \mathbf{x}, \nu), \end{aligned} \tag{78}$$

from where we define

$$\kappa_{in} = N_{scat} \int d^2\hat{n}'(\hat{n}' \cdot \hat{n})\sigma_{scat}(\hat{n} \to \hat{n}'), \tag{130}$$

where $\sigma_{scat}(\hat{n} \to \hat{n}')$ is the differential cross section for scattering of a photon traveling in a direction $\hat{n}$ into a direction $\hat{n}'$ and N_{scat} is the number of scatterers per gram. (These integrals are independent of the unit vector because of the invariance of the integrands under simultaneous rotations of $\hat{n}$ and $\hat{n}'$).

We can also express opacity as a function of the temperature and density proportional to powers of both:

$$\kappa(\rho, T) = \kappa_1 \rho^{\alpha} (k_B T)^{\beta}, \tag{131}$$

where κ_1, α and β are independent from the temperature and the density. The goal now is to relate α and β to actual physical processes and see how these results can be used to relate observable properties of stars.

Notes about the charge of the electron
The statcoulomb (statC) or franklin (Fr) or electrostatic unit of charge (esu) is the physical unit for electrical charge used in the esu-cgs (centimetre–gram–second system of units) and Gaussian (non rationalized) units. It is a derived unit given by

$$1statC = 1dyn^{1/2}cm = 1cm^{3/2}g^{1/2}s^{-1}.$$

That is, it is defined so that the Coulomb constant becomes a dimensionless quantity equal to 1. It can be converted using

$$1newton = 10^5 dyne$$

$$1cm = 10^{-2}m$$

The SI system of units uses the coulomb (C) instead. The conversion between C and statC:

$$1C \leftrightarrow 2997924580statC \approx 3.0010^9 statC$$

$$1statC \leftrightarrow \sim 3.3356410^{-10}C.$$

Thomson Scattering

Thomson scattering, which is the simplest contribution to opacity, is the elastic scattering of photons with energies much less than $m_e c^2$ on free electrons moving non-relativistically. The differential scattering cross section is

$$\sigma_{Thomson}(\hat{n} \to \hat{n}') = \frac{e^4}{2m_e^2 c^4}\left[1 + (\hat{n}' \cdot \hat{n})^2\right]. \tag{132}$$

This differential cross section is even in $\hat{n}'$ while the factor in Eq. (130)

$$\kappa_{in} = N_{scat} \int d^2\hat{n}'(\hat{n}' \cdot \hat{n})\sigma_{scat}(\hat{n} \to \hat{n}'), \qquad (130)$$

is odd in $\hat{n}'$, we get here $\kappa_{in} = 0$. (*Note from Weinberg's book: This forward–backward symmetry can be understood in classical terms. Classically, in Thomson scattering the electron position oscillates under the influence of the electric field of the incoming photon, and this oscillation produces the electromagnetic field of the outgoing photon. This oscillation is in the direction of the polarization vector of the incoming photon, which is normal to the photon's direction, so there is nothing about this oscillation or the field it produces that can distinguish between the forward and backward directions*.)

In consequence, where the opacity is dominated by Thomson scattering, the total opacity is

$$\kappa = \kappa_{out} = N_e \sigma_T \qquad (133)$$

where σ_T is the total Thomson scattering cross section given by the integral of the differential cross section (132) over the solid angle:

$$\sigma_T = \frac{8\pi}{3} \left(\frac{e^2}{\hbar c}\right)^2 \left(\frac{\hbar}{m_e c}\right)^2 = 0.66525 \times 10^{-24} \text{cm}^2 \qquad (134)$$

and N_e is the number of free electrons per gram. For a medium consisting of completely ionized atoms of atomic number Z and atomic weight A, we have $N_e = Z/Am_1$, where $m_1 = 1.66054 \times 10^{-24}$ g is the mass for unit atomic weight.

About unit atomic mass
The atomic mass (m_a or m) is the mass of an atom. Although the SI unit of mass is the kilogram (symbol: kg), atomic mass is often expressed in the non-SI unit atomic mass unit (amu) or unified mass (u) or dalton (symbol: Da), where 1 amu or 1 u or 1 Da is defined as $1/12$ of the mass of a single carbon-12 atom, at rest. The protons and neutrons of the nucleus account for nearly all of the total mass of atoms, with the electrons and nuclear binding energy making minor contributions. Thus, the numeric value of the atomic mass when expressed in daltons has nearly the same value as the mass number. Conversion between mass in kilograms and mass in daltons can be done using the atomic mass constant

$$m_{\mathrm{u}} = \frac{m(^{12}\mathrm{C})}{12} = 1\ \mathrm{Da}$$

The formula used for conversion is:

$$1\ \mathrm{Da} = m_{\mathrm{u}} = \frac{M_{\mathrm{u}}}{N_{\mathrm{A}}} = \frac{M(^{12}C)}{12\ N_{\mathrm{A}}} = 1.660\ 539\ 066\ 60(50) \times 10^{-24}\ \mathrm{g},$$

where M_u is the molar mass constant, N_A is the Avogadro constant and $M(^{12}\mathrm{C})$ is the experimentally determined molar mass of carbon-12.

This gives a Thomson scattering opacity (133) $\sigma_T = 0.400 \times Z/A\ \mathrm{cm}^2/\mathrm{g}$. Since the cross section is constant (aside from a possible dependence of the degree of ionization on temperature and density) the opacity for Thomson scattering has

$$\alpha = \beta = 0 \tag{135}$$

No averaging over photon frequency is necessary if Thomson scattering dominates the opacity.