Introduction to Astrophysics
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1.1 Radiative Models

We will be deriving the differential equations and boundary conditions for a star in which energy transport is dominated by radiation. We will see that for a set of stars of a given age and initial uniform chemical composition (ie with stars in many clusters), any stellar parameter, such as radius, luminosity, etc., may be expressed as a function of stellar mass. In consequence, when any two of these parameters are plotted against one another, the plot is a one-dimensional curve. (For example displaying the luminosity vs effective temperature, the Hertzsprung–Russell relation diagram).

With the chemical composition fixed and uniform, we can regard the pressure $p(r)$, opacity $\kappa(r)$, and nuclear energy production per mass $\epsilon(r)$ as fixed functions of the density $\rho(r)$ and temperature $T(r)$. The star's structure is then described by four functions of r: the mass $\mathcal{M}(r)$ contained within a sphere of radius r; the radiant energy per second $\ell(r)$ flowing outward through a spherical surface of radius r; and the density $\rho(r)$ and temperature $T(r)$. These four quantities are governed by four first-order differential equations: the equations of hydrostatic equilibrium (6)

$$\frac{dP(r)}{dr} = -G\frac{\rho(r)\mathcal{M}(r)}{r^2} \tag{6}$$

and (7)

$$\frac{d\mathcal{M}(r)}{dr} = 4\pi r^2 \rho(r) \tag{7}$$

and also the equations of radiative transport (101)

$$\frac{d\mathcal{L}(r)}{dr} = 4\pi r^2 \epsilon(r)\rho(r), \tag{101}$$

and (107)

$$\frac{dT(r)}{dr} = -\frac{3\kappa(r)\rho(r)\mathcal{L}(r)}{4acT^3(r)4\pi r^2}. \tag{107}$$

The pressure p, Rosseland mean opacity κ, and nuclear energy production per mass ϵ are assumed to be given as functions of density and temperature. We have then four equations for four unknown functions, $\rho(r)$, $\mathcal{M}(r)$, $T(r)$, and $\mathcal{L}(r)$. With appropriate boundary conditions we can attempt to solve them. We can take:

$$\mathcal{M}(0) = \mathcal{L}(0) = 0 \tag{110}$$

at the center of the star, and at the surface $r = R$:

$$\rho(R) = T(R) = 0 \tag{111}$$

We know that $T(R) = 0$ at the star surface is not compatible with observations. But practically we could use (110) as a good approximation and apply conditions like (111) at any other radius (this would be called "nominal radius"). In particular smoothness and isotropy in density breaks down at values of r for which $R - r$ is no longer large compared with the typical photon free path $1/\rho(r)\kappa(r)$ at r. This region is what is called the *stellar atmosphere*. In this region we would need to use equation (43) of radiative equilibrium. This nominal radius R (which is not the radius of the star's surface) is where the density and temperature would vanish if equations (6), (7), (101) and (107) are valid up to it.

For a real star we can define a surface at which the density and temperature vanish, a real "surface" with radius R_{true} beyond which there is mostly empty space, with only outgoing radiation and some gas at very low density, what is called the solar corona. But this is not the surface from which comes the light we see. To the extent that the light of a star resembles black-body radiation, we can think of it as coming from an effective surface with radius R_{eff}, defined by the equation

$$\sigma T^4(R_{eff}) \times 4\pi R_{eff}^2 = L, \tag{112}$$

$\sigma = \frac{2\pi^5 k^4}{15c^2h^3} = ac/4 = 1.3806503 \times 10^{-23}\ J/K$ is the Stefan–Boltzmann constant, $a = 4\sigma/c = 7.565767 \times 10^{-16}\ J/(m^3K^4)$ is called the radiation constant, L is the star's luminosity, which is actually the value of $\mathcal{L}(r)$ at any radius r outside the stellar core in which nuclear energy production occurs. The depth of the effective surface below the true surface can be characterized by what is called its optical depth,

$$\tau_{eff} = \int_{R_{eff}}^{R_{true}} \kappa(r)\rho(r)dr \tag{113}$$

The optical depth is a dimensionless number (κ has dimensions of $cm^2/grams$ and ρ has dimensions of $grams/cm^3$ and of course the radius has dimensions of length (i.e. cm). Since the typical photon free path is precisely $1/\kappa(r)\rho(r)$, we expect the optical depth τ_{eff} to be of the order one. It is customary to take it to be $2/3$, performing calculations that Weinberg call dubious in his book.

Obviously the thickness of the stellar atmosphere is much smaller than R the radius of the star. We could apply the differential equations (6), (7), (101) and (107) with the boundary conditions (110) and (111), and the understanding that for (112) we are only thinking that the temperature T and the density ρ are much lower than the corresponding values deep in the star's interior. For example, the central density and temperature of the Sun are $(\rho = 98 \pm 15)\ g/cm3$ and $T = (13.6 \pm 1.2) \times 10^6\ K$, while even deep in the stellar atmosphere, at an optical depth $\tau = 10$, the solar density and temperature are only about $\rho = 5 \times 10^{-7}\ g/cm^3$ and $T = 9,700K$.

Equations (6), (7), (101) and (107) are four order equations that depend only on one independent variable: the radius. The solutions then can be expressed as a one-parameter family of solutions. This is called:

Vogt-Russell theorem
The Vogt–Russell theorem states that the structure of a star, in hydrostatic and thermal equilibrium with all energy derived from nuclear reactions, is uniquely determined by its mass and the distribution of chemical elements throughout its interior. In all truthfulness this is not really a theorem because it has not being proven. We have two set of boundary conditions $\mathcal{M}(0) = \mathcal{L}(0) = 0$ at the center of the star, and at the surface $\rho(R) = T(R) = 0$ But we could just use $\rho(0)$ and $T(0)$ with some tentative trial values ρ_c and T_c at the center and have with $\mathcal{M}(0) = \mathcal{L}(0) = 0$ four initial conditions. When integrating (6), (7), (101) and (107) with these four initial conditions we can obtain a solution that will depend on ρ_c and T_c. We could adjust the values of ρ_c and T_c so that $\rho(R) = T(R) = 0$ at some radius R. And we will have a solution, but it might not be unique. These solutions will depend on the radius.

There are good reasons to choose as the free parameter the mass instead of the radius. For most of the life of the star the mass will not change much (it will start blowing mass only at an old age).

We can put then the equations in term on $\mathcal{M}$ starting with (6),

$$\frac{dr(\mathcal{M})}{d\mathcal{M}} = \frac{1}{4\pi r^2(\mathcal{M})\rho(\mathcal{M})} \tag{114}$$

(7)

$$\frac{dP(\mathcal{M})}{d\mathcal{M}} = -\frac{G\mathcal{M}}{4\pi r^4(\mathcal{M})}, \tag{115}$$

(101)

$$\frac{dP(\mathcal{L})}{d\mathcal{M}} = \epsilon(\mathcal{M}), \tag{116}$$

and (107) becoming

$$\frac{dT(\mathcal{M})}{d\mathcal{M}} = -\frac{3\kappa(\mathcal{M})\mathcal{L}(r)}{4acT^3(\mathcal{M})\left(4\pi r^2(\mathcal{M})\right)^2}. \tag{117}$$

We will know impose the following boundary conditions. At $\mathcal{M} = 0$,

$$r(\mathcal{M}) = \mathcal{L}(\mathcal{M}) = 0 \tag{118}$$

and at $\mathcal{M} = M$,

$$\rho(\mathcal{M}) = T(\mathcal{M}) = 0 \tag{119}$$

There is no need to input any other stellar parameter. It is this dependence of stellar structure on just the stars' mass that explains a striking result from observations of groups of stars (like clusters).

The dozens or hundreds of stars in the Pleiades (an open cluster) generally condensed at about the same time from the same cloud of interstellar material. This implies that they all have a near identical chemical composition and age as well as distance from us, but they do differ in their masses. The only observable distinctive feature of the stars in such a cluster are the stars' masses.

The main result is that when any pair of observables parameters for the cluster stars are plotted against each other, the resulting points for each star fall on a one-dimensional curve, each one corresponding to a different stellar mass.

This is not so pronounced for the thousands or hundreds of thousands of stars in a globular cluster like M15, where there is a greater spread in

Class	Effective temperature[1][2]	Vega-relative chromaticity[3][4][a]	Chromaticity (D65)[5][6][3][b]	Main-sequence mass[1][7] (solar masses)	Main-sequence radius[1][7] (solar radii)	Fraction of all main-sequence stars[8]
O	≥ 30,000 K	blue	blue	≥ 16 $M_\odot$	≥ 6.6 $R_\odot$	~0.00003%
B	10,000–30,000 K	blue white	deep blue white	2.1–16 $M_\odot$	1.8–6.6 $R_\odot$	0.13%
A	7,500–10,000 K	white	blue white	1.4–2.1 $M_\odot$	1.4–1.8 $R_\odot$	0.6%
F	6,000–7,500 K	yellow white	white	1.04–1.4 $M_\odot$	1.15–1.4 $R_\odot$	3%
G	5,200–6,000 K	yellow	yellowish white	0.8–1.04 $M_\odot$	0.96–1.15 $R_\odot$	7.6%
K	3,700–5,200 K	light orange	pale yellow orange	0.45–0.8 $M_\odot$	0.7–0.96 $R_\odot$	12.1%
M	2,400–3,700 K	orange red	light orange red	0.08–0.45 $M_\odot$	≤ 0.7 $R_\odot$	76.45%

Figure 1: Spectral lines, effective temperatures, luminosity and colors of stars(credit Wikipedia).

age and initial chemical composition. But even in an old cluster like this the plot of any pair of observables against each other is a more or less thickened curve.

The most easily observable stellar quantities are the luminosity L (or, if the distance d to the cluster is not known, the apparent luminosity $L/4\pi d2$) and the effective temperature T_{eff}.

The effective temperature is defined by the condition that $L = \sigma T_{eff}^4 \times 4\pi R^2$ but it is estimated from observations of the star's color and/or spectrum, as described in the figures 1 and 2.

Stellar Taxonomy

The history of stellar classification is a very interesting one. Something that has been recognized only lately is that many female astronomers were the leaders in the field and played a major role in its development.

Stellar classification started soon after Fraunhofer discovered the 574 dark lines in the solar spectrum. stellar classification is the classification of stars based on their spectral characteristics. Electromagnetic radiation from the star is analyzed by splitting it with a prism or diffraction grating into a spectrum exhibiting the rainbow of colors interspersed with spectral lines. Each line indicates a particular chemical element or molecule, with the line strength indicating the abundance of that element. The spectral class of a star is a short code primarily summarizing the ionization state, giving an objective measure of the photosphere's temperature.

Most stars are currently classified under the Morgan–Keenan (MK) system using the letters O, B, A, F, G, K, and M, a sequence from the hottest (O type) to the coolest (M type). Each letter class is then subdivided using a numeric digit with 0 being hottest and 9 being coolest (e.g., A8, A9, F0, and F1 form a sequence from hotter to cooler). The sequence has been expanded with classes for other stars and star-like objects that do not fit in the classical system, such as class D for white dwarfs and classes S and C for carbon stars.

In the MK system, a luminosity class is added to the spectral class using Roman numerals. This is based on the width of certain absorption lines in the star's spectrum, which vary with the density of the atmosphere and so distinguish giant stars from dwarfs. Luminosity class 0 or Ia+ is used for hypergiants, class I for supergiants, class II for bright giants, class III for regular giants, class IV for sub-giants, class V for main-sequence stars, class sd (or VI) for sub-dwarfs, and class D (or VII) for white dwarfs. The full spectral class for the Sun is then G2V, indicating a main-sequence star with a surface temperature around 5,800 K.

Type	*Lines*	T_{eff} (K)	*Color*	*Example*
O	HeII abs	>30,000	Sky blue	λ Ori
B	HeI abs, H	10,000–30,000	Blue–White	Rigel
A	H, CaII	7,500–10,000	White	Sirius A, Vega
F	CaII, H weaker	6,000–7,500	Yellow–White	Procyon
G	CaII, Fe, H weak	5,000–6,000	Yellow	Sun
K	Metals, CH, CN	3,500–5,000	Orange	Arcturus
M	TiO	<3,500	Red	Antares

Figure 2: Chromaticity, effective temperatures, colors and examples of stars from Weinberg's book.

The graph of observed absolute or apparent luminosity versus effective temperature is known as the Hertzsprung–Russell diagram.

In practice, the Hertzsprung–Russell diagram of a cluster is more like a thick curve than a straight or curved line. The reason for the dispersion is due to the different stage in the life of stars that one can plot at a given time, i.e. the dispersion in the initial conditions. But it can clearly be observed in the data that there is a one-dimensional curve of luminosity versus effective temperature, not points scattered all over the plot."

The Hertzsprung–Russell diagram for a cluster or group of stars commonly contains a main sequence, consisting of stars like the Sun that are still burning hydrogen at their cores. On the main sequence L increases smoothly with T_{eff}, with the most massive stars the hottest and most luminous.

As the cluster evolves, the Hertzsprung–Russell diagram develops a red giant branch, consisting of stars that have converted most of the hydrogen at their cores to helium, and are burning hydrogen only in a shell around the inert helium core. On this branch, the effective temperature

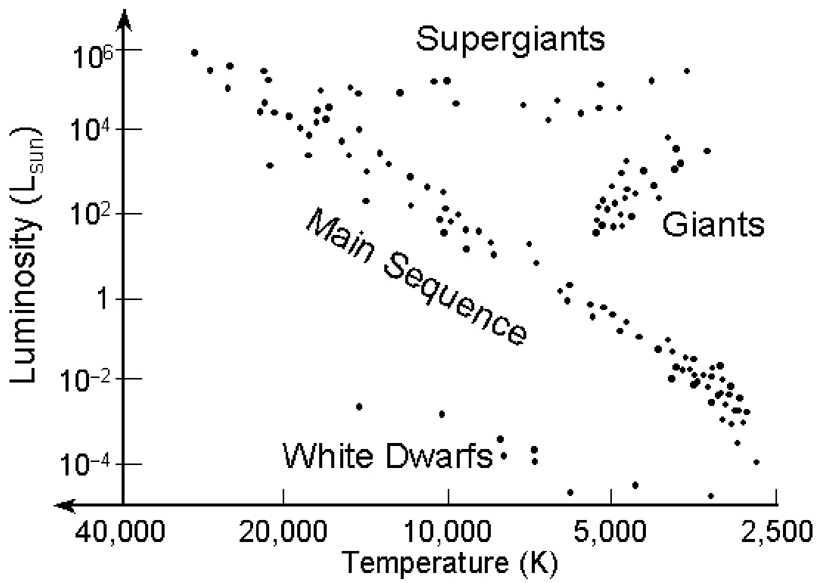

Figure 3: A sketch of an HR diagram for a few stars in our solar neighborhood (credit Prof. Richard Pogge, OSU, http://www.astronomy.ohio-state.edu/ pogge/Ast162/Unit1/hrdiag.html)

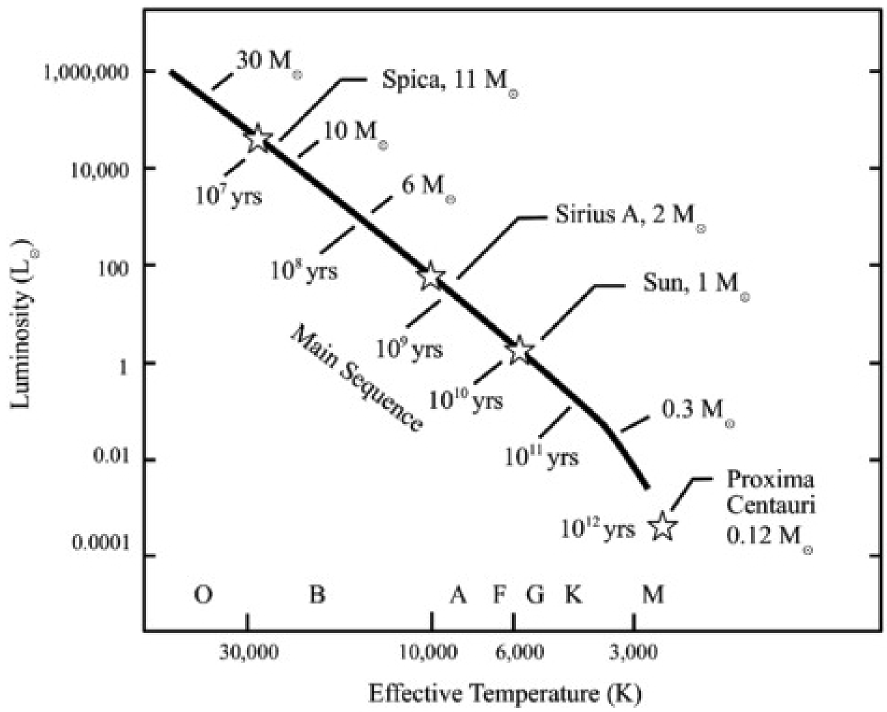

Figure 4: The relation between a star's luminosity, in units of the Sun's luminosity denoted $L_{\odot}$, and the star's effective temperature in kelvin degrees, for the main-sequence stars in the Hertzsprung-Russell diagram. The stellar masses that are given along the main-sequence curve are in units of the Sun's mass, denoted $M_{\odot}$. Stars of higher mass are hotter and more luminous. All of these stars shine by hydrogen burning with a lifetime that is also denoted along the main-sequence curve. More massive stars burn their hydrogen fuel at a faster rate and have shorter lifetimes. Copyright 2010, Professor Kenneth R. Lang, Tufts University

decreases (and radius increases) with increasing luminosity, accounting for the red color of very luminous red giant stars such as Betelgeuse and Antares. The heavier stars on the main sequence have larger L and therefore evolve more quickly, so as time passes more and more of the upper part of the main sequence bends over into the red giant branch. Observations of this main sequence turn-off therefore indicate the age of the cluster. Eventually the more massive stars of the cluster will begin to burn helium, and the Hertzsprung–Russell diagram will develop further complications, but it remains a more-or-less one-dimensional curve, as described by the Vogt–Russell theorem.

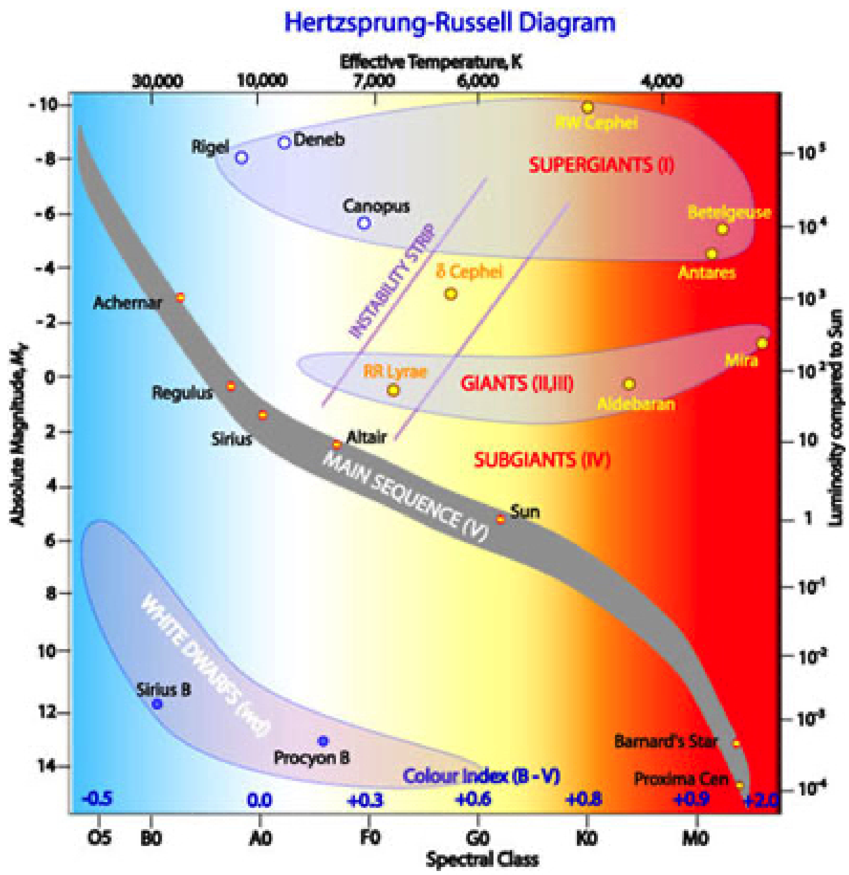

Figure 5: A typical H-R diagram from introductory textbooks