Introduction to Astrophysics
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For the temperature we have

$$k_B T(0) \cong M^C N_1^{C_1} N_2^{C_2}. \tag{198}$$

where

$$C = 2\frac{1+\lambda+\alpha}{3\lambda+\nu+3\alpha+\beta}, \tag{199}$$

and

$$C_1 = -\frac{1}{3\lambda+\nu+3\alpha+\beta}, \tag{200}$$

and

$$C_2 = 1 + 4C_1, \tag{201}$$

We are now in a position to understand the Hertzsprung–Russell relation between effective surface temperature and luminosity for stars on the main sequence based on first principles. Let's remember that from $L = 4\pi R^2 \sigma T_{eff}^4$ we have that dimensionally

$$k_B T_{eff} \equiv [k_B^4 L/4\pi\sigma R^2]^{1/4} = [k_B^4 L/4\pi ac R^2]^{1/4} = [N_1 L^*/\pi R^2 \kappa_1]^{1/4} \tag{202}$$

From this and from $R \propto M^A$ and $L \propto M^B$ we get that

$$T_{eff} \propto M^{[B-2A]/4} \tag{203}$$

We can define the "Hertzsprung-Russell" relationship as

$$L \propto T_{eff}^H \tag{204}$$

with

$$H = \frac{4B}{B-2A} = 4\left[1 - 2\frac{-2+\lambda+\nu+\alpha+\beta}{(1+\lambda+\nu)(3\alpha+\beta)+(3-\alpha-\beta)(3\lambda+\nu)}\right]^{-1} \tag{205}$$

The estimate of H is simplest for stars on the upper part of the main sequence, whose high temperature means that opacity is dominated by Thomson scattering, for which $\alpha = \beta = 0$. For both the proton–proton chain and the CNO cycle $\lambda = 1$, and then ν is a free parameter, so H is

$$H = \frac{12(3+\nu)}{11+\nu} \tag{206}$$

In all cases ν is positive-definite and $3.27 < H < 12$. For the proton–proton chain we have roughly $\nu \approx 5$ and , $H \approx 6$ and for the CNO cycle $\nu \approx 15$ and $H \approx 8.3$. The comparison with observation is complicated by the fact that, although it is straightforward to measure L for any star whose distance is known (or to measure ratios of values of L for a cluster of stars that are all at the same distance), it is difficult to obtain a precise value for T_{eff} from observations of colors or spectral lines.

From one graph of L versus T_{eff} for a large sample of stars with masses between 2 and 10 solar masses, Weinberg estimated that in the plot from figure 1 $H \approx 7$. The problems associated with the measurement of effective surface temperature can be avoided by considering the class of eclipsing binary stars. In this case accurate values of R and M can be found from the analysis of the time-dependence of luminosities and Doppler shifts. It is particularly revealing to consider the relation between luminosity and mass for stars, such as those on the upper part of the main sequence, whose opacity is due to Thomson scattering, for which $\alpha = \beta = 0$. For these stars Eqs. (194)–(196) give $B = 3,\ B_1 = -1,\ \text{and}\ B_2 = 4$, so here Eq. (197) reads

$$L \cong \epsilon_1 M^3 N_1^{-1} N_2^4 = \frac{ca(4\pi G m_1 \mu)^4}{\kappa_1 k_B^4} M^3. \tag{207}$$

Interesting enough this result is completely independent of the parameters $\epsilon_1,\ \lambda,\ \text{and}\ \nu$ which characterizes the nuclear reaction process. At this point it is really interesting to remark that Weinberg suspects that for $\alpha = \beta = 0$ (Thomson scattering) this result is even independent of the assumption that

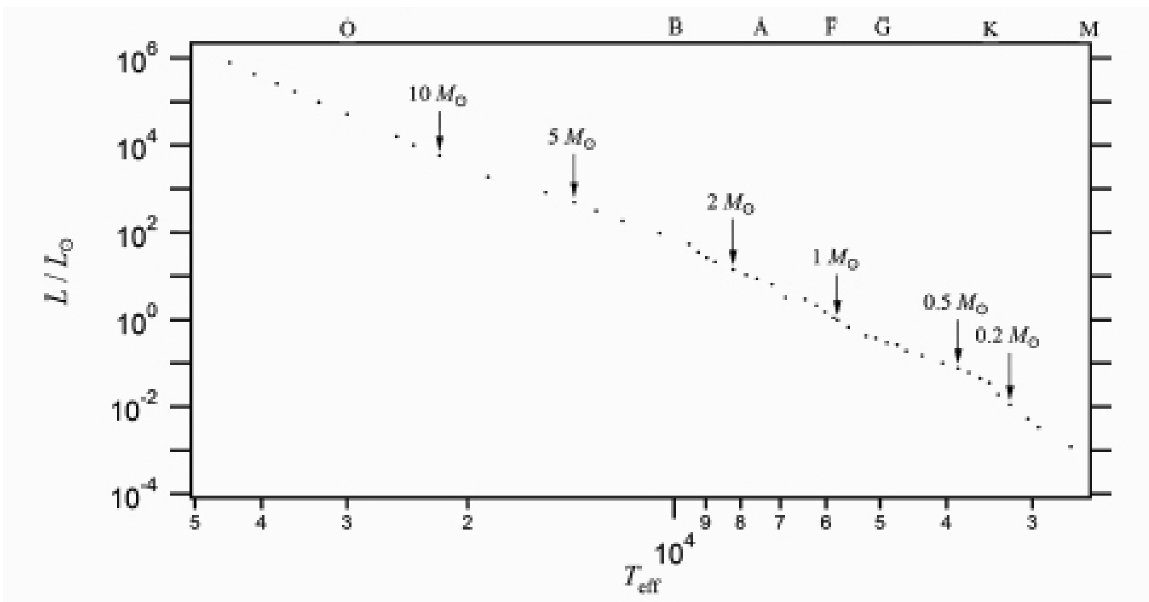

Figure 1: The plot shows main - sequence stars of various masses within an H–R diagram. The sample was taken among the 1000 nearest stars, obtained from the Gleise star catalogue. From LeBlanc, Introduction to Astrophysics

the rate of energy generation per mass is proportional to a product of powers of density and temperature, but he confess he has not been able to prove it. The data on eclipsing binaries cited by Andersen (Astron. Astrophys. Rev. 3, 91, 1991) shows that for $2 \leqslant M/M_{\odot} \leqslant 20$, binaries have $L \propto M^{3.6}$. A survey cited in J. P. Cox and R. T. Giuli, Principles of Stellar Structure (Gordon and Breach, New York, 1968), p. 15, shows that bright stars have $L \propto M^{4.0}$, while dimmer stars have $L \propto M^{2.76}$. Stars on the upper part of the main sequence according to C. J. Hansen, S. D. Kawaler, and V. Trimble, Stellar Interiors: Physical Principles, Structure and Evolution, 2nd edn. (Springer, New York, 2004), p. 28 have $L \propto M^{3.5}$. The limited statistics from eclipsing binaries and the oversimplification assuming that opacity is entirely due to Thomson scattering, explains the discrepancies among these measured exponents – 3.6, 4.0, 2.76, and 3.5 – and the theoretical result that $L \propto M^3$. The luminosity–mass relation provides insight regarding the scale of time over which stars of various mass evolve. The fusion $4^1H \rightarrow^4 He$ yields 6.5 MeV per proton, so the energy per mass available from hydrogen burning is

$$6.5 MeV/p \times 1.602 \times 10^{-5} erg/MeV/1.672 \times 10^{-24} g/p = 6.23 \times 10^{19} erg/g.$$

The Sun has mass $M_{\odot} = 1.93910^{33}$ g, but initially only 75% was hydrogen, so the energy available is

$$\begin{aligned} E_H &= 0.75 f \times 1.939 \times 10^{33} \left(\frac{M}{M_{\odot}}\right) g \times 6.23 \times 10^{19} \mathrm{erg/g} \\ &= 0.93 \times 10^{53} f \left(\frac{M}{M_{\odot}}\right) \mathrm{erg} \end{aligned}$$

where f is the fraction of the Sun's hydrogen that becomes sufficiently hot to initiate nuclear reactions. The Sun has luminosity $L_{\odot} = 3.84510^{33}$ erg/sec, so a star of mass M and luminosity L could go on burning hydrogen for a time

$$E_H/L \simeq 7.6 \times 10^{11} f \frac{M/M_\odot}{L/L_\odot} \text{years}$$

The main sequence duration of the Sun is commonly estimated as 10^{10} years, corresponding to $f \simeq 0.013$, a not unreasonable value. Even with an efficiency this small, the solar main sequence lifetime is much longer than the Kelvin time 10^7 years over which the Sun could go on shining without nuclear reactions, and it is not much less than the present age 1.37×10^{10} years of the big bang. But with the analytic dimensional estimate $L \propto M^3$ and the same hydrogen burning efficiency, for $M = 100 M_\odot$ the main sequence duration would be only 10^6 years, while with the empirical relation $L \propto M^{3.5}$ the main sequence lifetime would be 10^5 years. Finally, consider the relation between stellar radii and masses. Recall that $R \propto M^A$, and for $\alpha = \beta = 0$ and $\lambda = 1$, Eq. (190) gives

$$A = \frac{-1 + \nu}{3 + \nu}.$$

If for the CNO cycle we take $\nu = 15$, then $R \propto M^{0.78}$. Data cited by A. Weiss, W. Hillebrandt, H.-C. Thomas, and H. Ritter, Cox and Giuli's Principles of Stellar Structure, 2nd edn. (Cambridge Scientific Publishers, Cambridge, 2004), p. 10, for stars with masses between 5 and 20 solar masses give $R \propto M^{0.78}$, while other data cited by C. J. Hansen, S. D. Kawaler, and V. Trimble, Stellar Interiors: Physical Principles, Structure and Evolution, 2nd edn. (Springer, New York, 2004), p. 28 for stars on the upper part of the main sequence indicate that $R \propto M^{0.75}$. This is a very satisfactory confirmation of the results of dimensional analysis.

Radiation pressure

In this case $p = aT^4/3$ equation (179) and (180) become

$$\frac{d\left(\rho(r)k_BT(r)\right)}{dr} = -N_2\frac{\rho(r)\mathcal{M}(r)}{4\pi r^2} \qquad (179)$$

$$\frac{d\mathcal{M}(r)}{dr} = 4\pi r^2\rho(r) \qquad (180)$$

become

$$\frac{d\left(k_BT(r)\right)^4}{dr} = -3N_2'\frac{\mathcal{M}(r)\rho(r)}{4\pi r^2} \qquad (179)$$

and

$$N_2' \equiv 4\pi Gk_B^4/a, \qquad (208)$$

while the equations (175), and (176)-(178) will not change,

$$\epsilon = \epsilon_1\rho^\lambda(k_BT)^\nu, \qquad \kappa = \kappa_1\rho^\alpha(k_BT)^\beta, \qquad (175)$$

$$\frac{d\mathcal{L}^*(r)}{dr} = 4\pi r^2\rho^{\lambda+1}(r)\left(k_BT(r)\right)^\nu, \qquad (176)$$

$$\frac{d\left(k_BT(r)\right)^4}{dr} = -3N_1\rho^{\alpha+1}(r)\left(k_BT(r)\right)^\beta\frac{\mathcal{L}^*(r)}{4\pi r^2}, \qquad (177)$$

with

$$N_1 \equiv \frac{\kappa_1\epsilon_1k_B^4}{ca}. \qquad (178)$$

Now stellar parameters $R,\ L^* \equiv L/\epsilon_1,\ \rho(0),\ k_B T(0)$, etc. depend only on $N_1,\ N_2',\ \text{and}\ M$. Note that N_2' has the dimensions of $G[energy]^4/[energy/volume]$, or

$$
\begin{aligned}
N_2' &= [G][energy]^4/[energy/volume] = G][energy]^3[volume] \\
&= [length]^{12}[time]^{-8}[mass]^2.
\end{aligned} \tag{209}
$$

If the opacity is dominated by the Thomson scattering we have,

$$[L] = [energy]/[time] = [length]^2 = [length]^2[time]^{-3}[mass],$$

and then $L^* \equiv L/\epsilon_1$ has dimensions

$$[L^*] = [length]^{-3\lambda+2\nu}[time]^{-2\nu}[mass]^{1+\lambda+\nu}.$$

The following is the only possible combination of $N_1,\ N_2',\ \text{and}\ M$ to get

$$L = \epsilon_1 L^* \cong \epsilon_1 M^{B'} N_1^{B'_1} {N'_2}^{B'_2} \tag{210}$$

where

$$B' = 1 + \frac{(\lambda+\nu/2)(3\alpha+\beta) - (\alpha+\beta/2)(3\lambda+\nu)}{3\lambda+\nu+3\alpha+\beta}, \tag{211}$$

and

$$B'_1 = -\frac{(3\lambda+\nu)}{3\lambda+\nu+3\alpha+\beta}, \tag{212}$$

and

$$B'_2 = \frac{\nu}{4} + B'_1\left(-1 + \frac{\nu}{4} + \frac{\beta}{4}\right). \tag{213}$$

If we have $\alpha = \beta = 0$ in Eq (175) we get $B' = B'_2 = -B'_1 = 1$, and then Eq (210) becomes

$$L =\cong \epsilon_1 M N_1^{-1} N'_2 = \frac{4\pi Gc}{\kappa_1} M, \tag{214}$$

We should remember that in section where we study radiative models we found the Eddington limit for the luminosity.

$$\kappa(R) L < 4\pi GcM \tag{124}$$

and if the gas pressure was negligible compared with the radiation pressure (which only happens in very massive stars) we get from $dp_{gas}/dr \sim 0$

$$L = \frac{4\pi GcM}{\kappa(R)} \tag{125}$$

If we assume that the opacity is independent of temperature and density, as it is for Thomson scattering, then $\kappa(R)$ is the same as what in this section we have called κ_1, so Eq. (214) is the same as Eq. (125) and dimensional analysis is not needed. If we have a more general dependence of the opacity on temperature and density in equation (175) (i.e. $\alpha \neq 0 \neq \beta$) then Eq (125) will still hold but dimensional analysis may be useful.

Convection

Convection can be an efficient mechanism for transporting energy in stars. Hot buoyant mass elements can carry excess energy outward while cooler elements fall inward creating eddy currents. Suppose that a small element of stellar fluid happens to move upward from r to $r + dr$. The balance of forces at the surface of the element will cause the pressure inside to change, from $p(r)$ to the ambient pressure $p(r)+p'(r)dr$ at a different point.

We can assume that the process will be adiabatic because het conduction is very slow in stars. So ρ and T will be some function of the pressure. The new density will be

$$\rho(r) + \left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r)dr, \tag{215}$$

Assuming that variations are adiabatic if this new density is greater than the ambient density $\rho(r) + \rho'(r)dr$ at the new position, then the fluid element will sink back toward its original position, and the initial configuration will be stable. Thus the condition for stability against upward motion is

$$\left[\frac{\partial\rho(p)}{\partial p}\right]_{p=p(r)} p'(r) > \rho'(r). \tag{216}$$

Similarly, if the blob density (215) is less than the new ambient density $\rho(r) + \rho'(r)dr$ then the fluid element will float upward, so we then have stability against downward motion. Since for downward motion dr is negative, the stability condition is also (216).

On the other hand, if the left-hand side of Eq. (216) is less than the right-hand side we have an exponentially growing instability, whereas if the two sides are equal we have an unstable equilibrium (i.e an instability against a steady drift upwards or downwards.

Under conditions of convective stability, the r-derivative of the temperature is given by the equation (107) of radiative energy transport,

$$\frac{dT(r)}{dr} = -\frac{3\kappa(r)\rho(r)\mathcal{L}(r)}{4acT^3(r)4\pi r^2}. \tag{107}$$

while the r-derivative of the pressure is given by the equation (6) of hydrostatic equilibrium,

$$\frac{dP(r)}{dr} = -G\frac{\rho(r)\mathcal{M}(r)}{r^2} \tag{6}$$

so it is convenient (and conventional) to rewrite the equation (216) of convective stability in terms of temperature and pressure rather than density and pressure. Using the ideal gas law

$$\rho = mp/k_B T,$$

where m is the mass of the gas particles. We get,

$$\left[\frac{\partial \rho}{\partial p}\right] = \frac{\rho}{p} - \frac{\rho}{T}\left[\frac{\partial T}{\partial p}\right] = \frac{\rho}{p} - \left[\frac{\partial \ln T}{\partial \ln p}\right]$$

where we intentionally write the equations in a manner that we don't need to worry about the actual mass of the particles involved. The square brackets indicate adiabatic variations. The r variations are given by

$$\rho' = \frac{\rho p'}{p} - \frac{\rho T'}{T}$$

The quantities involved in (216) can then be written

$$\left[\frac{\partial \rho(p)}{\partial p}\right]_{p=p(r)} p'(r) - \rho'(r) = -\frac{p'(r)\rho(r)}{p(r)}\left(\nabla_{ad}(r) - \nabla(r)\right),$$

where

$$\nabla_{ad}(r) \equiv \left[\frac{\partial \ln T(p)}{\partial \ln p}\right]_{p=p(r)}, \tag{217}$$

and $\nabla(r)$ is the actual value of this derivative in the star:

$$\nabla(r) \equiv \frac{T'(r)/T(r)}{p'(r)/p(r)}. \tag{218}$$

Since the quantity $p'\rho/p$ is everywhere negative, the condition (216) for convective stability is just

$$\nabla(r) < \nabla_{ad}(r). \tag{219}$$

Using equation (6) and (107) we get

$$\nabla(r) = \frac{3\kappa(r)\mathcal{L}(r)p(r)}{16\pi caT^4(r)G\mathcal{M}(r)}. \tag{220}$$

It is useful to write the stability condition $\nabla(r) < \nabla_{ad}(r)$ as a limit on the rate of stable energy flow through a sphere of radius r that can be carried by radiation:

$$\mathcal{L}(r) < 4\nabla_{ad}(r)\left(\frac{p_{rad}(r)}{p(r)}\right)\mathcal{L}_{Edd}(r), \tag{221}$$

where $p_{rad}(r)$ is the radiation pressure $aT^4(r)/3$ and $\mathcal{L}_{Edd}(r)$ is the Eddington limit $4\pi GcM(r)/\kappa(r)$. As we saw at the end on the section on radiative models, $\mathcal{L}(r)$ must in any case be less than $\mathcal{L}_{Edd}(r)$ in order for radiation not to overcome gravitational attraction and tear the star apart. We will see that $4\nabla_{ad}$ is never very different from unity, so for ordinary stars, for which radiation pressure is much less than gas pressure, stability against convection requires that $\mathcal{L}(r)$ must be not just less but very much less than the Eddington limit $\mathcal{L}_{Edd}(r)$.

To calculate ∇_{ad} we make use of the conservation of energy and mass. (For relativistic theories, in which mass is not conserved, we use baryon number instead.) We take E as the thermal energy density, excluding the energy associated with rest masses, so the thermal energy per gram is E/ρ. When the volume per gram $1/\rho$ of stellar material increases by a small amount $\delta(1/\rho)$ (which of course is negative for decreasing volume), the work per gram that is done against the ambient pressure p is $p\delta(1/\rho)$, so in the absence of heat flow the conservation of energy requires that

$$\delta(\mathcal{E}/\rho) + p\delta(1/\rho) = 0 \tag{222}$$

We saw that when the energy is proportional to the pressure in the gas a polytropic equation is used to write $\mathcal{E} = \mathcal{E}(p)$

$$\mathcal{E} = \frac{p}{\Gamma - 1} \tag{223}$$

also written as $\mathcal{E} = np$ where $n \equiv 1/(\Gamma - 1)$

Using (223) the condition for the conservation of energy (222) in an adiabatic process becomes

$$\Gamma p \delta(1/\rho) + (1/\rho)\delta p = 0 \tag{224}$$

in compact form

$$\delta(p/\rho^{\Gamma}) = 0 \tag{225}$$

The partial derivative in (216) is now

$$\left[\frac{\partial \rho(p)}{\partial p}\right] = \frac{\rho}{\Gamma p}, \tag{226}$$

so the stability condition expressed by (216) is now by the condition

$$\frac{\rho(r)p'(r)}{\Gamma p(r)} > \rho'(r), \tag{227}$$

Multiplying both sides by $\Gamma/\rho(r)$

$$\frac{p'(r)}{p(r)} > \frac{\Gamma \rho'(r)}{\rho(r)}. \tag{228}$$

The quantity

$$\frac{p'(r)}{p(r)} - \frac{\Gamma \rho'(r)}{\rho(r)}. \tag{229}$$

is called the Schwarzschild discriminant.