Introduction to Astrophysics
Fall 2021
October 13, 2021

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Crossover

The temperature at which the rates of energy production in the CNO cycle and proton–proton chain equalize is called **crossover temperature**. The rate of the reactions in the proton–proton chain is suppressed by the Coulomb barrier by a factor $\exp(-15.7 \times (10^7 T)^{-1/3})$ while then the rate of the reactions in the CNO cycle is suppressed by a factor $\exp(4.4 \times -15.7 \times (10^7 T)^{-1/3})$. It is further suppressed relative to the proton–proton chain by the ratio of the number of CNO nuclei to hydrogen nuclei, which for the Sun is about 10^{-3}, and since a photon is emitted, also by a factor $e^2/\hbar c \simeq 10^{-2}$. On the other hand, the reaction $p + p \to d + e^+ + \nu$ in the proton–proton chain is a weak interaction, so its rate is proportional to the square of the weak coupling constant, and is therefore suppressed by a dimensionless factor $(G_{wk}E^2)^2$, which for $E \sim 1$ MeV is about 10^{-22}. So, very roughly, the crossover temperature at which the CNO cycle and the proton–proton chain have competitive rates is given by

$$10^{-13} \times 10^{-2} \times \exp(-4.4 \times 15.7(10^7 T)^{-1/3}$$
$$\approx 10^{-22} \times \exp(-15.7(10^7 T)^{-1/3}) \approx 2.5 \times 10^7 \ \text{ K},$$

This is higher than the estimated central temperature of the Sun (1.36×10^7 K). Notice then that this means that for stars that have masses lower than the $M_\odot$ the proton-proton chain dominates energy production while with masses higher than the Sun starting with those which have core temperatures $T > 2.5 \times 10^7$ K the CNO cycle dominates it.

Beyond Hydrogen Burning

When the hydrogen has been converted to helium in a star's center, the star leaves the main sequence and becomes a red giant, in which the conversion of hydrogen to helium continues in a shell surrounding the helium core. The core temperature continues to grow, and when it becomes sufficiently high it becomes the turn of helium to undergo nuclear reactions. Although there is no stable nucleus that can be formed in a collision of a proton and a ^4He nucleus or in the collision of two ^4He nuclei, in particular this one can produce an unstable state of the nucleus of ^8Be that lives long enough before it undergoes fission back into two ^4He nuclei, so that it can serve as an intermediary in the carbon production reactions:

$$\begin{aligned} \mathrm{a}: \quad & ^4He +^4 He \to^8 Be + \gamma \\ \mathrm{b}: \quad & ^4He +^8 Be \to^{12} C + \gamma. \end{aligned} \tag{174}$$

This sequence of two-body reactions, does not lead to an energy production rate per volume $\epsilon\rho$ proportional to ρ^2, as in the proton–proton chain and the CNO cycle. This is because there is only a small probability $\mathcal{P}$ for the ^8Be nucleus to absorb another ^4He nucleus before it fissions. Thus $\epsilon\rho$ is proportional to $\rho^2\mathcal{P}$, and since $\mathcal{P}$ when small is itself proportional to ρ, $\epsilon\rho$ is proportional to ρ^3, and therefore the exponent λ in Eq. (1.5.1) is $\lambda = 2$. The temperature dependence of ϵ is harder to estimate. The first step in the reaction a is endothermic, requiring an energy E of relative motion of the two ^4He nuclei of at least 92 keV. In order for ^4He nuclei to have any chance of having energies this large, the temperature must be at least 10^8 K. Even at such relatively high temperatures, there are sizable Coulomb barriers both in the rate for reaction a and in the probability $\mathcal{P}$ that a ^8Be nucleus will experience reaction b instead of fissioning. The only reason why carbon production is non-negligible at temperatures of order 10^8 K to 10^9 K is that there is an unstable state of ^{12}C that provides a resonance in the $^4He +^8 Be$ channel at an accessible excitation energy of $310ZZ$ keV. This unstable state has an appreciable chance of decaying into the stable ground state of carbon, with the emission of a 7.4 MeV photon. Because of the pair of Coulomb barriers plus the exothermic nature of reaction a, the exponent ν in equation (148)

$$\epsilon(\rho, T) \simeq \epsilon_1 \rho^\lambda (k_B T)^\nu, \qquad (148)$$

for the temperature dependence of carbon production is quite large, estimated to be of order 30 to 40, depending on the temperature. Once ^{12}C is formed in this way, it is possible to produce heavier nuclei in various reactions that are suppressed mostly by Coulomb barriers: $^4He +^{12} C \to^{16} O + \gamma$, $^4He +^{16} O \to^{24} Mg + \gamma$, $^{12}C +^{12} C \to^{24} Mg + \gamma$, and so on. There are also reactions that destroy but do not produce various light nuclei with relatively small binding energies, including ^2H, ^3He, ^6Li, ^7Li, ^9Be, ^{10}B, and ^{11}B. These nuclei are found spectroscopically in interstellar clouds, and in consequence their measured abundance provides a valuable lower bound on the cosmological abundance of light elements left over from the beginning of the big bang.

The main sequence

We saw that stellar parameters such as radius, luminosity, central temperature, effective surface temperature, etc. all depend only on the star's mass, age, and initial chemical composition. This is why, when any pair of these parameters for a sample of stars in a cluster that all began at the same time with the same uniform chemical composition are plotted against each other, the values of these parameters will fall close to a one-dimensional curve, such as the Hertzsprung–Russell diagram comparing luminosity and effective surface temperature. We are interested in understanding the functional relationship among them. IN general plotting the curves requires detailed physical assumptions and numerical calculation. For stars that are still on the main sequence, burning hydrogen at their cores, it is possible to make a good estimate of the form of these curves using dimensional analysis, together with the assumption of power-law behavior for the rate per mass ϵ of nuclear energy generation and for the opacity κ:

$$\epsilon = \epsilon_1 \rho^\lambda (k_B T)^\nu, \qquad \kappa = \kappa_1 \rho^\alpha (k_B T)^\beta, \tag{175}$$

with κ_1, ϵ_1, α, β, λ, ν all constants which we assume to depend only on the chemical composition. (ie. before we found that $\alpha = \beta = 0$ for Thomson scattering, and $\alpha = 1$ and $\beta = -7/2$ for free-free absorption. Later we found $\lambda = 1$ for the p-p chain and CNO cycle; $\nu \approx 5$ for the p-p chain and a larger value for the CNO cycle, and also ν weakly depending on the temperature, $\nu \propto T^{-1/3}$).

Our analysis will be limited to stars in which thermal energy is transported only by radiation. Each stellar parameter will turn out to be dependent only on the star's mass M and a pair of quantities N_1 and N_2 that depend on chemical composition and fundamental physical constants. Since there are no dimensionless ratios among $M,\ N_1,\ \text{and}\ N_2$, any stellar parameter will be proportional to a product of powers of $M,\ N_1,\ \text{and}\ N_2$, with exponents fixed by dimensional analysis. This only works for stars on the main sequence whose chemical composition (on which κ_1, α, etc. depend) is still approximately uniform. For red giant stars whose stellar parameters also depend on the radius of the helium core, dimensional analysis is not enough. It is also not enough even if we assume that non-uniformities evolve from an initially uniform composition, because then stellar parameters depend on the age of the star, as well as on $M,\ N_1,\ \text{and}\ N_2$.

We will write the equations for the luminosity and temperature (101) and (107)

$$\frac{d\mathcal{L}(r)}{dr} = 4\pi r^2 \epsilon(r) \rho(r), \tag{101}$$

$$\frac{dT(r)}{dr} = -\frac{3\kappa(r)\rho(r)\mathcal{L}(r)}{4acT^3(r)4\pi r^2}. \tag{107}$$

in terms of $\rho,\ k_BT,$ and $\mathcal{L}^* \equiv \mathcal{L}/\epsilon_1$;

$$\frac{d\mathcal{L}^*(r)}{dr} = 4\pi r^2 \rho^{\lambda+1}(r)\left(k_BT(r)\right)^{\nu}, \tag{176}$$

$$\frac{d\left(k_BT(r)\right)^4}{dr} = -3N_1\rho^{\alpha+1}(r)\left(k_BT(r)\right)^{\beta}\frac{\mathcal{L}^*(r)}{4\pi r^2}, \tag{177}$$

with

$$N_1 \equiv \frac{\kappa_1\epsilon_1 k_B^4}{ca}. \tag{178}$$

We will begin by assuming that the pressure p is dominated by gas pressure, which is the case for all but the most massive stars. The pressure then is well approximated by the ideal gas law, $p = k_BT\rho/m_1\mu$, where μ is the molecular weight and m_1 is the mass of unit atomic weight. And the equations (6) and (7) become

$$\frac{d\left(\rho(r)k_BT(r)\right)}{dr} = -N_2\frac{\rho(r)\mathcal{M}(r)}{4\pi r^2} \tag{179}$$

$$\frac{d\mathcal{M}(r)}{dr} = 4\pi r^2\rho(r) \tag{180}$$

where

$$N_2 \equiv 4\pi G m_1\mu. \tag{181}$$

If we are dealing with uniform chemical composition, the stellar parameters $R,\ \mathcal{L}^* \equiv \mathcal{L}/\epsilon_1,\ \rho(0),\ k_BT(0)$, etc. can depend only on N_1, N_2, and M. Now we want to determine the proper dimensionalities of N_1 and N_2 in terms of length, time and mass. Energy production rate has dimensions,

$$[\epsilon] = [energy][mass]^{-1}[time]^{-1} = [velocity]^2[time]^{-1} = [length]^2[time]^{-3},$$

$$[\epsilon_1] = [length]^2[time]^{-3}[mass/length^3]^{-\lambda}[energy]^{-\nu},$$
$$= [length]^{2+3\lambda-2\nu}[time]^{-3+2\nu}[mass]^{-\lambda-\nu}. \quad (182)$$

Since $1/\kappa\rho$ is the mean free path opacity it has dimensions $[\kappa] = [length]^{-1}/[mass/length^3]$ and then

$$[\kappa_1] = [length]^{-1}[mass/length^3]^{-1-\alpha}[energy]^{-\beta},$$
$$= [length]^{2+3\alpha-2\beta}[time]^{2\beta}[mass]^{-1-\alpha-\beta}. \quad (183)$$

and

$$[ca/k_B^4] = [energy][time]^{-1}[area]^{-1}[energy]^{-4}$$
$$= [energy]^{-3}[length]^{-2}[time]^{-1},$$
$$= [length]^{-8}[time]^5[mass]^{-3}. \quad (184)$$

and N_1 and N_2

$$[N_1] = [length]^{12+3\lambda-2\nu+3\alpha-2\beta}[time]^{-8+2\nu+2\beta}[mass]^{2-\lambda-\nu-\alpha-\beta} \quad (185)$$

$$[N_2] = [G][mass] = [velocity]^2[length] = [length]^3[time]^{-2}. \quad (186)$$

To calculate the stellar radius R, we ask what product of the expression $M^A N_1^{A_1} N_2^{A_2}$ has the dimensions of length. Setting the numbers of powers of length, time, and mass in this product respectively equal to +1, 0, and 0, we find

$$\text{powers of length}: \ 1 = (12 + 3\lambda - 2\nu + 3\alpha - 2\beta)A_1 + 3A_2, \quad (187)$$

$$\text{powers of time}: \ 0 = (-8 + 2\nu + 2\beta)A_1 - 2A_2, \quad (188)$$

$$\text{powers of mass}: \ 0 = A + (2 - \lambda - \nu - \alpha - \beta)A_1, \tag{189}$$

Using Eq. (188) to eliminate A_2 in Eq. (187) gives A_1; Eq. (188) then gives A_2; and using this in Eq. (189) gives A.

$$A = \frac{-2 + \lambda + \nu + \alpha + \beta}{3\lambda + \nu + 3\alpha + \beta}, \tag{190}$$

Then

$$A_1 = \frac{1}{3\lambda + \nu + 3\alpha + \beta}, \tag{191}$$

and

$$A_2 = \frac{-4 + \nu + \beta}{3\lambda + \nu + 3\alpha + \beta}, \tag{192}$$

The stellar radius R then will be

$$R \cong M^A N_1^{A_1} N_2^{A_2} \tag{193}$$

with A, A_1, and A_2 provided by (190)-(192). For the luminosity we get

$$[L] = [energy]/[time] = [length]^2[time]^{-3}[mass],$$

so $L^* \equiv L/\epsilon_1$ has the dimensions

$$[L^*] = [length]^{-3\lambda+2\nu}[time]^{-2\nu}[mass]^{1+\lambda+\nu}.$$

Then

$$L^* \cong M^B N_1^{B_1} N_2^{B_2}.$$

where

$$B = \frac{(1+\lambda+\nu)(3\alpha+\beta)+(3-\alpha-\beta)(3\lambda+\nu)}{3\lambda+\nu+3\alpha+\beta}, \tag{194}$$

and

$$B_1 = -\frac{(3\lambda+\nu)}{3\lambda+\nu+3\alpha+\beta}, \tag{195}$$

and

$$B_2 = \frac{\nu(3\alpha+\beta)+(4-\beta)(3\lambda+\nu)}{3\lambda+\nu+3\alpha+\beta}, \tag{196}$$

and obviously

$$L = \epsilon_1 L^* \cong \epsilon_1 M^B N_1^{B_1} N_2^{B_2}. \tag{197}$$