Introduction to GR-2020
Homework 2

Undergraduate students work through exercises 1 to 7, graduate students all 9 exercises:

## 1 Exercise 1

Given the numbers:
$\{A^0 = 5, A^1 = 0, A^2 = -1, A^3 = -6\}, \{B_0 = 0, B_1 = -2, B_2 = 4, B_3 = 0\}$, and,
$C = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 5 & -2 & -2 & 0 \\ 4 & 5 & 2 & -2 \\ -1 & -1 & -3 & 0 \end{pmatrix}$ find:

(a) $A^\alpha B_\alpha$;
(b) $A^\alpha C_{\alpha\beta}$ for all $\beta$;
(c) $A^\gamma C_{\gamma\sigma}$ for all $\sigma$;
(d) $A^\nu C_{\mu\nu}$ for all $\mu$;
(e) $A^\alpha B_\beta$ for all $\alpha, \beta$;
(f) $A^i B_i$;
(g) $A^j B_k$ for all $j, k$;

## 2 Exercise 2

Identify the free and dummy indices in the following equations and change them into equivalent expressions with different indices. How many different equations does each expression represent?
(a) $A^\alpha B_\alpha = 5$;
(b) $A^{\bar{\mu}} = \Lambda^{\bar{\mu}}_{\nu} A^\nu$;
(c) $T^{\alpha\mu\lambda} A_\mu C_\lambda{}^\gamma = D^{\gamma\alpha}$;
(d) $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = G_{\mu\nu}$;

## 3 Exercise 3

A collection of vectors $\{\vec{a}, \vec{b}, \vec{c}, \vec{d}\}$ is said to be linearly independent if no linear combination of them is zero except the trivial one, $0\vec{a} + 0\vec{b} + 0\vec{c} + 0\vec{d} = 0$.
(a)Show that the basis vectors $\vec{e}_0 = (1, 0, 0, 0), \vec{e}_1 = (0, 1, 0, 0), \vec{e}_2 = (0, 0, 1, 0), \vec{e}_3 = (0, 0, 0, 1)$

are linearly independent.
(b) Is this set of basis vectors $\{\vec{a}, \vec{b}, \vec{c}, 5\vec{a} + 3\vec{b} - 2\vec{c}\}$ LI?

## 4 Exercise 4

(a)Prove that the zero vector $(0, 0, 0, 0)$ has these same components in all reference frames.
(b)Use (a) to prove that if two vectors have equal components in one frame, they have equal components in all frames.

## 5 Exercise 5

Given $\vec{A}$ in system $\mathcal{O}$ as $(0, -2, 3, 5)$, find:
(a)the components of $\vec{A}$ in $\bar{\mathcal{O}}$, which moves at speed $0.8$ relative to $\mathcal{O}$ in the positive $x$ direction;
(b)the components of $\vec{A}$ in $\bar{\mathcal{O}}$, which moves at speed $0.6$ relative to $\mathcal{O}$ in the positive $x$ direction;
(c)the magnitude of $\vec{A}$ from its components in $\mathcal{O}$;
(d)the magnitude of $\vec{A}$ from its components in $\bar{\mathcal{O}}$;

## 6 Exercise 6

The following matrix gives a Lorentz transformation from $\mathcal{O}$ to $\bar{\mathcal{O}}$:

$$\begin{pmatrix} 1.25 & 0 & 0 & .75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ .75 & 0 & 0 & 1.25 \end{pmatrix}$$

(a)What is the velocity (speed and direction) of $\bar{\mathcal{O}}$ relative to $\mathcal{O}$?
(b)What is the inverse matrix to the given one?

## 7 Exercise 7

(a)Show that the sum of any two orthogonal spacelike vectors is spacelike.
(b)Show that a timelike vector and a null vector cannot be orthogonal.

### The following exercises are for the graduate students:

## 8 Exercise 8

(a)Find the energy, rest mass, and three-velocity $v$ of a particle whose four-momentum has the components $(4, 1, 1, 0)kg$ (units of mass and $c = 1$).
(b)The collision of two particles of four-momenta $\vec{p}_1 = (3, -1, 0, 0)kg$ and $\vec{p}_2 = (2, 1, 1, 0)kg$ results in the destruction of the two particles and the production of three new ones, two of which have four-momenta $\vec{p}_3 = (1, 1, 0, 0)kg$ and $\vec{p}_4 = (1, -1/2, 0, 0)kg$.
Find the four-momentum, energy, rest mass, and three-velocity of the third particle produced. Find the CM frames three-velocity.

## 9 Exercise 9

A body is said to be uniformly accelerated if its acceleration four-vector $\vec{a}$ has constant spatial direction and magnitude, say $\vec{a} \cdot \vec{a} = \alpha^2 \geqslant 0$.
(a) Show that this implies that always has the same components in the body's MCRF (momentarily comoving reference frame), and that these components are what one would call acceleration in Galilean terms. (This would be the physical situation for a rocket whose engine always gave the same acceleration.)
(b)Suppose a body is uniformly accelerated with $\alpha = 10ms^{-2}$ (about the acceleration of gravity on Earth). If the body starts from rest, find its speed after time t. (Be sure to use the correct units.) How far has it traveled in this time? How long does it take to reach v = 0.999?
(c)Find the elapsed proper time for the body in (b), as a function of t. (Integrate $d\tau$ along its world line.) How much proper time has elapsed by the time its speed is v = 0.999? How much would a person accelerated as in (b) age on a trip from Earth to the center of our Galaxy, a distance of about $2 \times 10^{20}$ m?